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On the spectra of finite group presentations in signal processing

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ABSTRACT
Spectral analysis plays an important role in the study or investigation of signal representation. This paper aimed at investigating the signal representation as decimation from algebraic structure using spectral decomposition of modules. The aim is achieved by considering mapping of linear transformations of a group G to a signal space X(n) using different techniques for spectral analysis.

KEYWORDS
finite group; representation; Fourier analysis; spectral analysis

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(1) BACKGROUND OF THE STUDY

Representation theory is a powerful tool for obtaining information about finite groups with applications in areas such as signal processing, cryptography, sound compression which depend on Fast Fourier Transform (FFT) for finite groups [1], [2]. Its emergence also serves as tool to obtain information about finite groups through the methods of linear algebra. Generally, Signal is the output of an array of sensor configured in time or space. It is a conveyor of information and whenever a signal is registered, it is processed to extract the information coded in it. Signals are also physical processes that spread over time which can be observed directly at the moment and form in which it appears. However, this is sometimes inconvenient and, in some cases, impossible. As such, one is always ultimately condemned to finiteness and discreteness in the observation and study of signals. Each signal can therefore be observed in a finite time interval irrespective of its duration, and its parameters can be measured and evaluated only discretely. Signals are also considered as functions on groups into fields or some discrete groups which is mostly identified with the group of integers Z or its subgroups Z_p of integer's modulo p. Hence, it is defined as a mapping $f: Z \to X$ or $f: Z_p \to Z_p$ where X could be the complex field C, real field R or some finite fields [2].

Based on the above concept, signal processing algorithm can be visualized as a mapping algorithm from a geometric structure of a group to its algebraic structure using linear transformation or invariants (see Lenz, [3]). This is a framework of group and representation theory. Thus, any signal processing algorithm requires a complete exploration of all possible symmetries in the signal. Since digital signals are finite dimensional, within the approximation of digitization, the signals exist in a vector space of finite basis. Rajathilagam *et al* [4] studied the existence of this basis in its vector space and then observed that one can get close to this finite basis from different perspective without any approximation using group theory. Also discussed by Riley, *et al*, [5], Knapp, [1], Duzhin *et al*, [6], Hamermesh, [2] and Mallat, [7], it was shown that group theory provides the actual piece of the puzzle which is the dimension of the vector space where the signal exists. It also shows the way for choosing specific transformation that is required for a particular application customized for specific characteristic.

Fourier transform was initially developed for continuous signals and then approximated for digital signals. In G-lets by Rajathilagam *et al*, [4], the most natural signal decomposition algorithm for a digitized signal without any artificially introduced approximation using general transformations was provided but avoided the use of representations and other properties of transformation group. Other literatures such as "The abstract harmonic analysis on finite, non-abelian groups" by Knapp, [1], lays the foundation in this paper due to the importance of spectral techniques and Fourier analysis in the consideration of various classes of signals [8]. In particular, the basic interest and approach in this paper is to focused on transfer of some important results from geometric structures to algebraic structures on the real line to produce different forms of signals. It is found that finite non-Abelian groups are more suitable for the purpose of this paper.

We defined some relevant terms from Fourier Analysis as follows: (see Benjamin, [9]).

Definition 1.1: (Periodic Function): A function $f: Z \rightarrow C$ is said to be periodic with period n if $f(x) = (x+n)$ for all $x \in Z$.

Definition 1.2: (Fourier Transform): Let $f: Z_n \rightarrow C$ define the Fourier Transform $\hat{f}: Z_n \rightarrow C$ of f by

$$\hat{f}(\overline{m}) = n\langle \chi_m, f \rangle = \sum_{k=0}^{n-1} e^{-2\pi imk} f(\overline{k}).$$

Definition 1.3: (Convolution): Let G be a finite group and $a,b \in L(G)$. Then the convolution $a * b: G \rightarrow C$ is defined by

$$a * b(x) = \sum_{y \in G} a(xy^{-1})b(y).$$

Hence, to each element $g \in G$, we can associate the delta function δ_g so that when a multiplication $*$ is assigned to L(G), then we get $\delta_g * \delta_h = \delta_{gh}$. Indeed

$$\delta_g * \delta_h(x) = \sum_{y \in G} \delta_g(xy^{-1})\delta_h(y)$$

and the only non-zero term is when $y = h$ and $g = xy^{-1} = xh^{-1}$, i.e. $x = gh$.

Now, if V is an inner product space, then a representation $\phi: G \rightarrow GL(V)$ is called *unitary* if ϕ_g is unitary for all $g \in G$, i.e., $\langle \phi_g(v), \phi_g(w) \rangle = \langle v, w \rangle$ for all $v,w \in W$. Identifying $GL_1(C)$ with C, we see that a complex number z is unitary if and only if $\bar{z} = z^{-1}$, that is $\bar{z}z = 1$. But this says exactly that $|z| = 1$, so $U_1(C)$ is exactly the unit circle S^1 in C. Hence a one-dimensional unitary representation is a homomorphism $\phi: G \rightarrow S^1$.

A simple example is a function $\phi: R \rightarrow S^1$ defined by $\phi(t) = e^{2\pi it}$. Then ϕ is a unitary representation of R since $\phi(t + s) = e^{2\pi i(t+s)} = e^{2\pi it}e^{2\pi is} = \phi(t)\phi(s)$ and produced a symmetric signal as follows:

```
>> t = linspace(0,10,20);
>> Y = (exp(2*pi*i*t));
>> stem(t,Y,'filled')
```

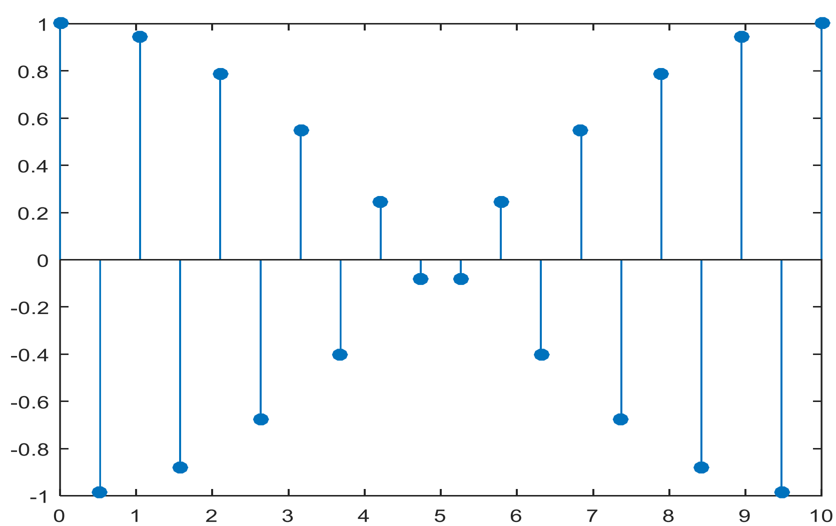


FIGURE 1.1: Symmetric signal produced by a unitary representation

It turns out that every representation of a finite group G is equivalent to a unitary representation. The result was also established by Benjamin, [9].

(2) FOURIER ANALYSIS ON FINITE GROUP

Fourier transform permits us to study and analyze the structure and the representation of finite groups [10]. Due to its vast applications in the area of signal and image processing, it is regarded as one of the most important aspects of mathematics [11], [12]. Most especially, the Fourier transform on cyclic groups. The idea is that values of $\hat{f}$ correspond to the wavelengths associated to the wave function f which is used to compress the wave. This idea can be used to produce and transform signals of different amplitude and frequency.

Fourier series also generate a representation for a signal in terms of Sine and Cosine functions, which gives an infinite set of basis vectors for the signal. In this case, time and frequency information cannot be localized on specific parts of the signal. Wavelets can also be used to create orthogonal subspaces using the transformations dilation and translation to capture different frequencies of the signal. But still due to the edge effect produced by breaking pieces, in application like face recognition they may not be able to testify a good match. Ridgelet transform as used by Candes, [13] and [14], only focuses on detecting lines in an image using Radon transform, see also in Matus, [15]. Orthonormal set of polynomials can also be generated by the orbit of a vector using representation theory as discussed by Vale, [16], Starck *et al*, [17], Aharom *et al*, [18]. It was discovered that combining more transformations as symmetry operations and their irreducible representations produced better output which is the main objective of this paper.

2.1 Signal Representation

The study and investigation of signal representation as decimation from the algebraic structure was given by Puchel, [19]. The author also presented the spectral decomposition of the module $X = C[\alpha]/(p(\alpha))$ (called polynomials) as the Fourier transform in ASP theory. If we let $p(\alpha) = \alpha^r - 1,$ then $X = C[\alpha]/(\alpha^n - 1)$ is a finite shift invariant signal model [19]. Again, from Fourier transform, the relationship between two signal spaces X and Y was obtained by Zhihai, [20]. If $a = (1, \alpha, \ldots, \alpha^{n-1})$ is a basis modules for X and $Y = C[\alpha]/(\alpha^{\frac{n}{m}} - 1)$ is another signal model whose basis modules are $a = (1, \alpha, \ldots, \alpha^{\frac{n}{m}-1}),$ $m,n,c \in Z^+$, $n = cm$, then from Fourier transform, if $\sigma \in X$ and $\lambda \in Y$ such that $\sigma = (\sigma(\alpha_1), \sigma(\alpha_2), \ldots, \sigma(\alpha_n))$ and $\lambda = (\lambda(\beta_1), \lambda(\beta_2), \ldots, \lambda(\beta_n))$, then $\lambda(\beta_k) = \sigma(\alpha_{ki})$, $0 \le i \le \frac{n}{m} - 1$. The Discrete Fourier Transform (DFT) used to convert a finite list of equally spaced sample signal into a list of coefficients of a finite combination of complex sinusoids, ordered by their frequencies. But the actual position of occurrence for component frequency in the given signal was never realized. Hence the drawback of this technique.

(3) METHODOLOGY

In this section, an algebraic structure on an operator group $L(G)$ coming from the convolution product is introduced. But Fourier transform can be used to analyze this structure more clearly in terms of rings as discussed by William, [8]. We first begin with the classical case of periodic functions on the set of integers. If n is a period for a function f, then so is any multiple of n and there exists a bijection between f and the elements of $L(Z_n)$, that is, functions $f: Z_n \to C$. The irreducible characters in this case form a basis for $L(Z_n)$ and Fourier transform encodes it as a function.

Example 3.1: (Periodic functions on Z). Let f,g: $Z \to C$ be periodic functions with period n. Then we defined their convolution as

$$f * g(m) = \sum_{r=0}^{n-1} f(m-r)g(r).$$

The Fourier transform is then

$$\bar{f}(m) = \sum_{r=0}^{n-1} e^{-2\pi imr/n} f(r).$$

Again, by Fourier inversion, we have

$$f(m) = \frac{1}{n}\sum_{r=0}^{n-1} e^{2\pi imr/n} \bar{f}(r).+$$

By multiplication formula, $\overline{f * g} = \bar{f} \cdot \bar{g}$.

Now, with G as defined above, if $\tau, \sigma \in G$, then for any operator δ, $\delta_\tau * \delta_\sigma = \delta_{\tau\sigma}$.

Note also that if $\lambda_1, \lambda_2 \in L(G)$, then define

$$\lambda_1 = \sum_{\tau \in G} \lambda_1(\tau)\delta_\tau \quad \text{and} \quad \lambda_2 = \sum_{\sigma \in G} \lambda_2(\sigma)\delta_\sigma.$$

Thus, since $L(G)$ is a ring, then the distributive law would yield

$$\lambda_1 * \lambda_2 = \sum_{\tau,\sigma \in G} \lambda_1(\tau)\lambda_2(\sigma)\delta_\tau * \delta_\sigma = \sum_{\tau,\sigma \in G} \lambda_1(\tau)\lambda_2(\sigma)\delta_{\tau\sigma}.$$

Applying the change of variables $x = \tau\sigma$, $y = \sigma$ gives

$$\lambda_1 * \lambda_2 = \sum_{x \in G} \left(\sum_{y \in G} \lambda_1(xy^{-1})\lambda_2(y) \right) \delta_x.$$

The above construction is built on the work of William, [8], on non-Abelian Harmonic Analysis and DSP Applications. He presented the basic non-Abelian group theory (for simple groups) relevant to Digital Signal Processing (DSP) and defined a generalized non-Abelian translation and its consequences and generalized convolution. The same method is being used in this paper for finite Abelian and non-Abelian groups.

- **Spectral Analysis**

According to Fourier Transform [8], since signal is composed of a set of component frequencies in the form of sine and cosine waves, each component frequency can be defined by $\zeta_j = j/\tau$, where τ is the time period of the signal and the component frequencies are given by the discrete Fourier series

$$\xi_n = \sum_{n=0}^{N-1} x_n e^{-i\zeta_j n/N} \qquad 3.1$$

Now, Fourier Transform has been defined in terms of group theory as a transformation group whose elements are rotations. The rotations in this case are represented by roots of unity in the complex field such that $\omega = 2\pi/\tau$, where τ is the time period of the signal. The generated cyclic group form the characters of the transformations which is used to build the Fourier transform basis in the frequency domain. From representation theory, characters of group form an orthogonal basis of a vector space X. If the binary operation of the group G is multiplication and we denote the group by $(G, \cdot)$, then G is the direct product of cyclic groups generated by the elements ω_r of order n^r, $r \in Z^+$, given by

$$G = \prod_{r=1}^{n} \langle \omega_r \rangle.$$

In this case, the elements of G are of the form $\omega_1\omega_2\ldots\omega_n$. Let $\omega, \sigma \in G$ such that $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. Then we define a group algebra over the vector space X as follows:

$$X[\omega] = \left\{ \sum_{r \in N} \zeta_r \sigma \mid \zeta_r \in X, r \in N \right\} = \left\{ \zeta(\omega) = \sum_{r \in N} \zeta_r \omega^{\sigma} \mid \zeta_r \in X, r \in N \right\},$$

with binary operation as multiplication defined by

$$\zeta(\sigma) \cdot \zeta'(\sigma) = \sum_{r \in N} \left(\sum_{s \in N} \zeta_s \zeta'_{r-s} \right) \omega^{\sigma}.$$

In order to investigate the signal representation as decimation from the algebraic structure, we consider the spectral decomposition of the modules $X = C[\alpha]/(p(\alpha))$ as Fourier transform, given in Section 2.1. Thus if the representation of the signals $x(n)$ and $y(n)$ are given by $x(n) = \sum_{i=1}^{n} x_i(\phi_i)$ and $y(n) = \sum_{i=1}^{n} y_i(\varphi_i)$ respectively, then the Fourier transform of $x(n)$ and $y(n)$ can be written as

$$x(\phi_k) = \sum_{i=1}^{n} x_i e^{-\frac{2\pi}{n} ijk}, \quad y(\varphi_k) = \sum_{i=1}^{n} y_i e^{-\frac{2\pi}{n} ijk}, \; 0 \le k \le n \qquad 3.2$$

Alternatively, the representation of $y(n)$ can be express as

$$y(n) = \sum_{i=1}^{n} y_i(\varphi_i) = \sum_{i=1}^{\frac{n}{2}-1} y_i(\varphi_i) + \sum_{i=\frac{n}{2}}^{n} y_i(\varphi_i) = \sum_{i=1}^{\frac{n}{2}-1} (y_i + y_{\frac{n}{2}+i})(\varphi_i) \qquad 3.3$$

so that the corresponding Fourier transform for $y(n)$ is

$$y(\varphi_k) = \sum_{i=1}^{\frac{n}{2}-1} (y_i + y_{\frac{n}{2}+k}) e^{-\frac{4\pi}{n} ijk}, \quad 0 \le k \le n/2 \qquad 3.4$$

(4) RESULT AND DISCUSSION

From Section 3 above, we assumed that elements of G are of the form $x(n)$. Thus, the operator $L(G)$ is considered as an operator over a signal space.

Now, define two special functions as follows:

i. A δ-function $\delta(n)$ defined by $\delta(n) = \begin{cases} 1 & if \quad n = 0, \\ 0 & if \quad n \neq 0. \end{cases}$

ii. A step function $\mu(n)$ defined by $\mu(n) = \begin{cases} 1 & if \quad n \ge 0, \\ 0 & if \quad n < 0. \end{cases}$

If $n = 0$, then $\mu(n) = \delta(n)$, if $n = 1$, then $\mu(n) = \delta(n - 1)$ and so on. Thus,

$$\mu(n) = \delta(n) + \delta(n - 1) + \delta(n - 2) + \ldots$$

i.e. $\mu(n) = \sum_{k=0}^{\infty} \delta(n-k)$ 4.1

Again, since $\delta(n) = 0$ for $n < 0$,

$$\mu(n) = \sum_{k=-\infty}^{n} \delta(k) \qquad 4.2$$

But $\mu(n) = \delta(n)$ if and only if $n = 0$. Thus, from equations 4.1 and 4.2,

$$\mu(n) = \sum_{k=-\infty}^{\infty} \delta(n-k) \qquad 4.3$$

Again, $L(x(n)) = y(n)$ for any transformation L on some $x(n) \in$ G which produces an output $y(n)$. Thus, by linearity,

$L(\alpha x(n)) = \alpha L(x(n))$;

$L(\alpha x(n) + \beta x(n)) = \alpha L(x(n)) + \beta L(x(n))$

where the coefficients α and β are constants. Thus, by equation 4.3, any discrete signal $x(n)$ can be expressed as

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \qquad 4.4$$

where the coefficients $x(k)$ can be real or complex. Hence, it follows from equation 4.4 that any signal $x(n)$ can be represented as a linear superposition of weighted and shifted impulse response (signals), also called signal basis.

Again, if

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k),$$

then

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)L(\delta(n-k)). \qquad 4.5$$

Moreover, by linear time-invariant, if $h(n)$ is any discrete time signal such that $L(x(n)) = h(n)$, then $L(x(n\text{-}k)) = h(n\text{-}k)$. Thus, if $L(\delta(n)) = h(n)$, then $L(\delta(n\text{-}k)) = h(n\text{-}k)$. Hence, from equation 4.5,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$= x(n) * h(n)$$

so that $y(n) \in G$ is expressed as a product $x(n) * h(n)$ for some $x(n), h(n) \in G$.

Example 4.1: Let x (-3n + 2) be any discrete-time signal and let δ, σ and μ be the transformations shifting, flipping and scaling respectively. Then

$\delta(n) = x(n + 2);$

$\sigma(n) = \delta(-n);$

$\mu(n) = \sigma(3n) = \delta(-3n) = x(-3n + 2).$

Thus, $x(-3n + 2)$ is decomposed by the transformations δ, σ and μ such that

$x(-3n + 2) = \delta(n) + \sigma(n) + \mu(n).$

4.1 Spectral Analysis on Finite Groups

Let G be a group with binary operation as defined above and let X be an n-dimensional space. Then for all $\upsilon, \omega \in X$, we have

$$X[\upsilon] = X[\omega] = \left\{ \zeta(\omega) = \sum_{r\in N} \zeta_r \omega^r \mid \zeta_r \in X, r \in N \right\} \quad \text{if and only if} \quad \omega_\tau = \upsilon_\tau^{-1} \quad \text{for} \quad \tau = 1,2,\ldots,n,$$

$\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n).$

Thus, from Equation 3.1, each $\zeta(n) \in (X, \pi)$ can be represented as $\zeta(n) = \sum_{r\in N} \zeta_r (n)^r \in X[n]$ and for any $q = (q_1, q_2, \ldots, q_n) \in X$, we have $\zeta(\xi^q) = \sum_{r\in N} \zeta_r \xi^{qr} = \sum_{r\in N} \zeta_r \xi_1^{q_1 r} \xi_2^{q_2 r} \ldots \xi_n^{q_n r}$.

Hence, given any $\tau, \sigma \in G$ and an element $\zeta(n) \in (X, \pi)$, we have

$$\sigma^\tau\left(\zeta(\xi^q)\right) = \sigma^\tau\left(\sum_{r\in N} \zeta_r \xi^{qr}\right) = \sigma^\tau\left(\sum_{r\in N} \zeta_r \xi_1^{q_1 r} \xi_2^{q_2 r} \ldots \xi_n^{q_n r}\right)$$

$$= \sum_{r\in N} \sigma^\tau(\zeta_r) \sigma^\tau(\xi_1^{q_1 r}) \sigma^\tau(\xi_2^{q_2 r}) \ldots \sigma^\tau(\xi_n^{q_n r})$$

$$= \sum_{r\in N} \zeta_r \xi_1^{q_1 r \tau^n} \ldots \xi_n^{q_n r \tau^n}.$$

Again if $\chi_q : G \to X$ is a mapping from the group G to the space X defined by

$$\omega_1^r \omega_2^r \ldots \omega_n^r \mapsto \zeta^{rq_1} \zeta^{rq_2} \ldots \zeta^{rq_n} = \zeta^{rq}, \; rq = r(q_1, q_2, \ldots, q_n),$$

then by linearity, we have

$$\sum_{r\in N} \zeta_r \omega^r \to \zeta(\xi^q) = \sum_{r\in N} \zeta_r \chi_q(\omega^r) = \sum_{r\in N} \zeta_r \xi_1^{rq_1} \xi_2^{rq_2} \ldots \xi_n^{rq_n} = \sum_{r\in N} \zeta_r \xi^{rq}$$

and taking all the χ_s for all $s \in N$, we have the Discrete Fourier Transform

$$\zeta(\omega) = \sum_{r\in N} \zeta_r (\omega^r) \mapsto \varphi(\zeta(w)) = \sum_{s\in N} (\sum_{r\in N} \zeta_r \xi^{rs}) \upsilon^s = \sum_{s\in N} \zeta(\xi^s) \upsilon^s.$$

Supposed now that $Y[n] = \{y(n) \in Y \mid \zeta(y(n)) = \sum_{s \in N} \zeta_s(\xi^s) y^s\}$, then

$$x(n) \cdot y(n) = \sum_{s \in N} \left(\sum_{r \in N} \zeta_s \xi^{s+r}) \right) \tau^s .$$

Again, for any $h \in N$,

$$\sum_{s \in N} \zeta_s \zeta_{s+h} = \sum_{s \in N} \zeta_s \sum_{c \in N} \zeta_c \xi^{c(s+h)} = \sum_{c \in N} \tau^c \xi^{ch} \sum_{s \in N} \zeta_s \xi^{cs} .$$

$$= \sum_{c \in N} \zeta_c \xi^{ch} \zeta(\xi^c)$$

so that

$$x(n) * y(n) = \sum_{h \in N} \left(\sum_{c \in N} \zeta_c \xi^{ch} \zeta(\xi^c) \right) x^h .$$

If we let $\sigma(n) = \sum_{c \in N} \zeta_c \zeta(\xi^c) x^c$, then it follows that

$$\varphi(\sigma(n)) = \sum_{c \in N} \sigma(\xi^c) y^c = \sum_{c \in N} \left(\sum_{s \in N} \zeta_c \xi^{cs} \zeta(\xi^c) \right) y^c .$$

Hence, $x(n) * y(n) = \varphi(\sigma(n))$.

If we define $x(n) \in X$ and $y(n) \in Y$ as above and let $x(n) = (x(\phi_1), x(\phi_1), \ldots, x(\phi_n))$, $y(n) = (y(\varphi_1), y(\varphi_1), \ldots, y(\varphi_n))$, then from Equations 3.2, 3.3 and 3.4, we have

$$x(\phi_{ki}) = \sum_{i=1}^{n} x_i e^{-\frac{4\pi}{n} ijk} = \sum_{i=1}^{\frac{n}{2}-1} x_i e^{-\frac{4\pi}{n} ijk} + \sum_{r=1}^{\frac{n}{2}-1} x_{r+\frac{n}{2}} e^{-\frac{4\pi}{n}(r+\frac{n}{2})k} = \sum_{i=1}^{\frac{n}{2}-1} (x_i + x_{k+\frac{n}{2}}) e^{-\frac{4\pi}{n} ijk} = y(\varphi_k) .$$

Hence, $y(\varphi_k) = x(\phi_{ki})$.

Again, let H be a cyclic subgroup of the group G. Then the representations of H can be used to produce a spiral signal (Signals that are bounded by elements of cyclic group) as follows: Since signals are regarded as functions on groups into fields and Discrete signals as signals on some discrete groups usually identified with the group of integers Z or its subgroups Z_p of integer's modulo p, if $\xi \in H$ is any representation of H, then we defined a function $\xi : Z \to X$ by $\xi(n) = \rho^{2k\pi/n}$ for all $n \in Z^+$, $k \geq 0$ where X could be the complex field C, the real field R, or some finite fields. Then ξ distribute every $n \in$Z over X as given in the figure below:

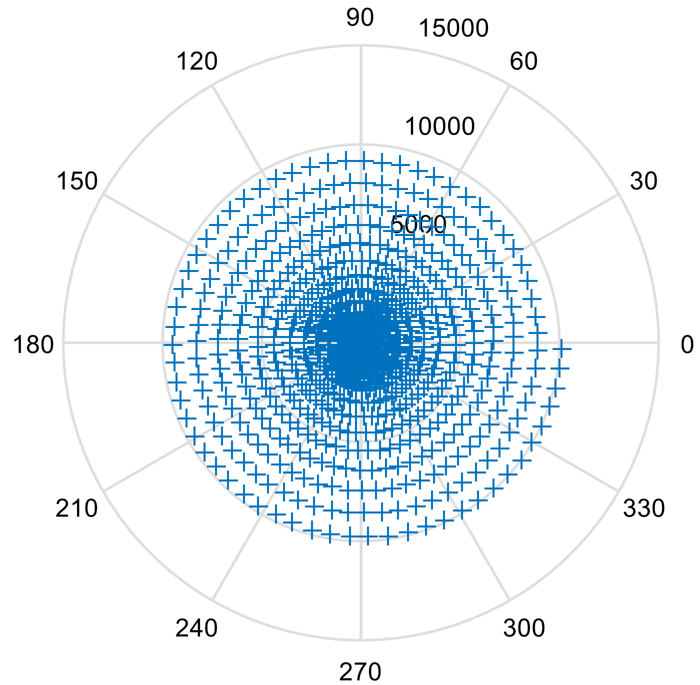


FIGURE 4.1: Path of signal generated by cyclic groups

The example 4.1 given above consist of the following transformations: **Flipping:** With the given function ξ, for every $n \in Z$, there exists $-n \in Z$ such that $\xi(x(n)) = x(-n)$. This transformation is called flipping. **Rotation:** There are n-rotation matrices in any Geometric group D_n and if the dimension of a signal is n, then the representation matrix of the linear transformation will be $n \times n$. The rotation matrix is then given by

$$\alpha_{\alpha_i} = \begin{pmatrix} \cos\alpha_i & -\sin\alpha_i \\ \sin\alpha_i & \cos\alpha_i \end{pmatrix}, \ i \in Z^+ \text{ and } x^{\tau_\alpha}(n) \mapsto x^{\tau_{2\alpha}}(n) \text{ for any } \alpha.$$

Thus, if a signal $x(n)$ is defined on a circle with rotation angle α, then the transformation $\xi \in G$ is defined by $x^{\xi}(n) = x(n-\alpha)$. **Scaling:** This demands that if ξ is a function whose domain is a set of transformations, then $\xi(x(n)) = x(cn)$ where c is a constant such that $c \in Z^+$. In this case, the signal is compressed by c-unit. Similarly, if $\xi(x(n)) = x(n/c)$, then the signal is delayed (or expanded) by c-unit. **Shifting:** If $x(n)$ is the original signal, then $x(n) \mapsto x(n-n_0)$ implies that the signal is shifted forward by n_0-unit and $x(n) \mapsto x(n+n_0)$ implies that the signal is shifted backward by n_0-unit.

For example, let $y(n) = x(-5n+3) \in (X, \pi)$ and σ, δ and τ be the transformations shifting, flipping and scaling respectively. Then $y(n)$ can be constructed as follows:

$$\mu(n) = \sigma(5n) = \delta(-5n) = x(-5n+3) \text{ such that } y(n) = \delta(n) + \sigma(5n) + \mu(n).$$

A simple example is shown in Figure 4.2 and 4.3 below. The original signal is given in Figure 4.2 and is splitted using other transformations whose sum gives the original signal as seen in Figure 4.3.

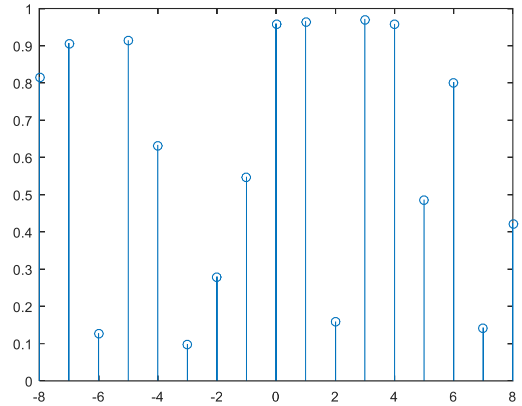


FIGURE 4.2: A single data series of a discrete-time signal

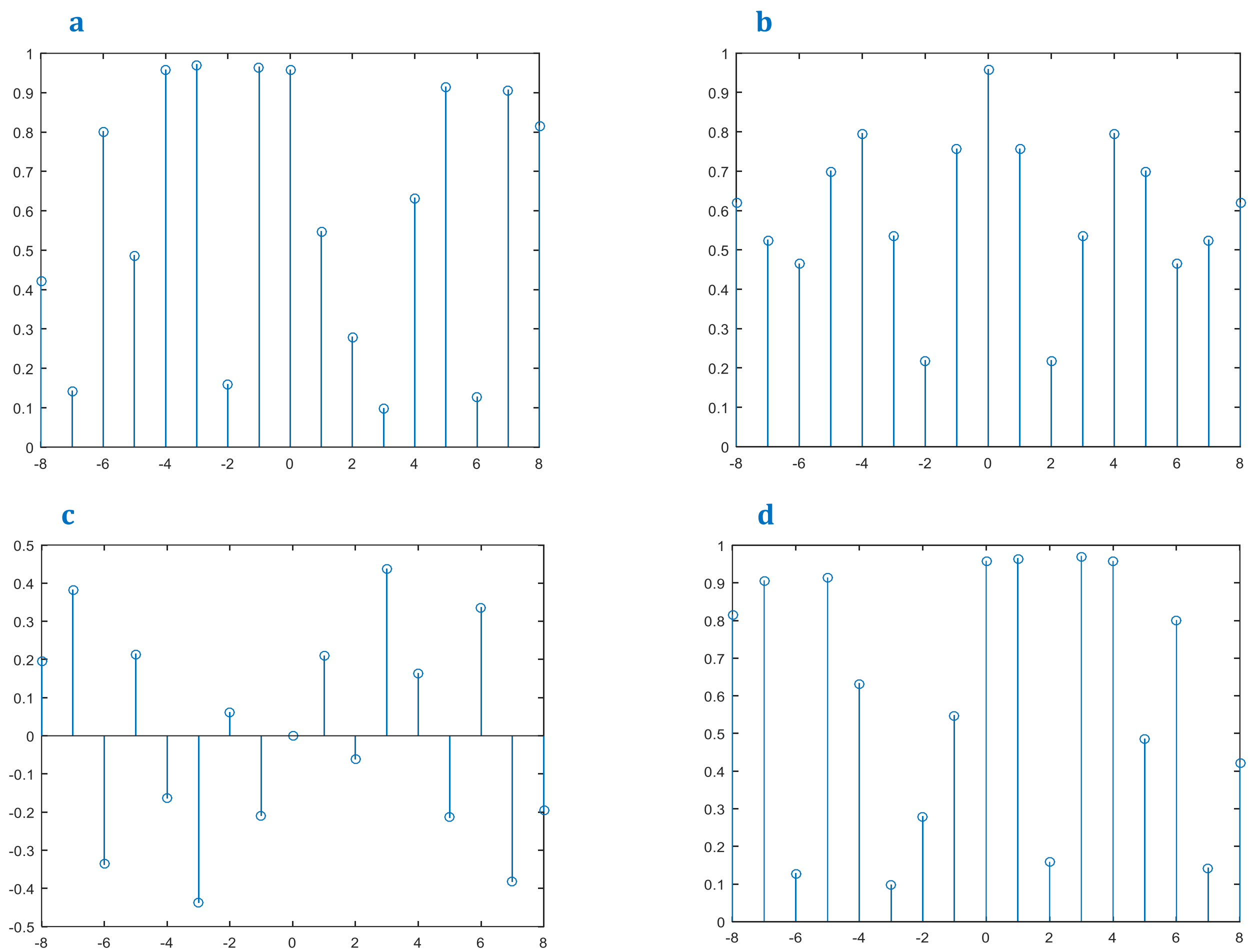
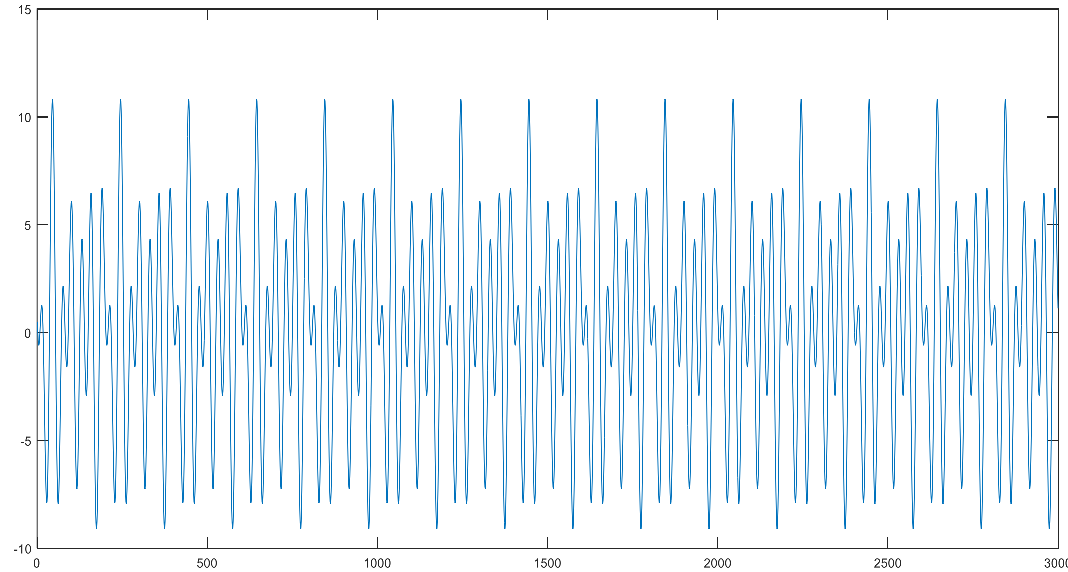


FIGURE 4.3: A subplots of Figure 4.2 using transformations

The spectral analysis is necessary because it gives detail information about a signal and help identify the impurities present in a signal. The basic idea behind Fast Fourier Transform (FFT) algorithm is to convert time domain signal into frequency domain signal and to determine the amplitude, frequencies and phases of the sinusoids present in a signal. Supposed now that the sampling frequency (fs) is 2000 with a sinusoid $x = a\cos(2\pi ft + \phi)$ where ϕ is the phase difference, let f = 40, 50, 70. Then the time domain signal is given in Figure 4.4.

a



b

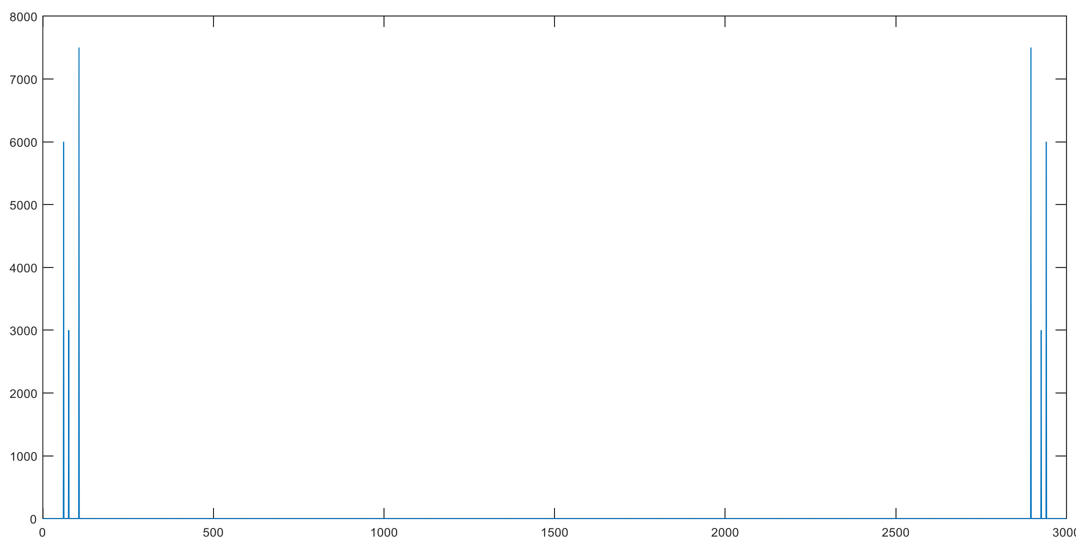


Figure 4.4: Converting time domain signal to frequency domain signal using FFT

The time domain signal (Figure 4.4a) is converted to frequency domain signal (Figure 4.4b) using the FFT. The disadvantage of the time domain signal is the difficulty in determining the frequency of a signal but with the help of FFT, the frequency, amplitude and phase of the sinusoids present in the signal can be determined. Finally, using Group theory, the mapping of linear transformations of a Group G to a signal may be done in different ways for spectral analysis.

4.2 Conclusion

The results in this paper provide a method of multiresolution analysis in terms of amplitude and frequency separately without an approximation. This method may be useful for edge detection, face detection, face recognition, denoising, object recognition, shape detection, text identification in images and other applications in image processing and robotics.

REFERENCES

[1] Knapp A. W.: Group Representations and Harmonic Analysis, Part II, *American Mathematical Society*, 1996; (43) pp.537-549.

[2] Hamermesh M.: Group Theory and Its Application to Physical Problems, New York, Dover Publications; 1989.

[3] Lenz R., and Thanh B.: Invariants: A Group Theoretical Foundation and some Applications in Signal Processing and Pattern Recognition. *International Symposium on Nonlinear Theory and its Applications* (NOLTA2005), Bruges, Belgium, 2005; pp.433-436.

[4] Rajathilagam B., Murali R. and Soman K.P.: "G-Lets: A New Signal Processing Algorithm", *Int. J. Comp. App.,* 2012; Vol. 37, No. 6, pp.0975-0987.

[5] Riley, K. F., Hobson, M. P., and Bence, S. J.: Mathematical Methods for Physics and Engineering, Cambridge University Press; 2002.

[6] Duzhin, S. V., and Chebotarevsky, B. D.: Transformation Groups for Beginners, Student Mathematical Library 25; 2004.

[7] Mallat S.: Geometrical grouplets, *Appl. Comp. Harmonic Analysis,* 2009; (26) pp.161-180.

[8] William J. D.: Topics in Nonabelian Harmonic Analysis and DSP Application, Proceedings of the International Symposium on Musical Acoustics, Nara, Japan; 2004.

[9] Benjamin S.: Representation theory of Finite Groups. Carleton University, Carleton; 2009.

[10] Dresselhaus, M. S.: Group Theory, Springer; 2008.

[11] Lenz R.: Group Theoretical Methods in Image Processing, Springer-Verlag New York, Inc.; 1990.

[12] Viana, M., and Lakshminarayanan, V.: Dihedral Fourier analysis, Lecture Notes in Statistics, Springer, New York, NY.; 2010.

[13] Candes, E.J.: Ridgelets: theory and applications, PhD thesis, Stanford University; 1998.

[14] Candes, E. J., and Donoho, D. L.: Ridgelets: The key to high dimensional intermittency, *Phil. Trans. R. Soc.* London A. 1999; (357) pp.2495-2509.

[15] Matus F. and Flusser J.: Image representations via a finite Radon transform, *IEEE Transactions on Pattern Analysis and Machine Intelligence,* 1993; (15) pp.996-1006.

[16] Vale, R., and Waldron, S.: Tight frames generated by finite non-abelian groups, Numerical Algorithms, 2008; (48) pp.11-27.

[17] Starck, J. L, Elad, M. and Donoho, D.: Image decomposition via die combination of sparse representation and a variational approach, *IEEE Trans. Image Processing,* 2005; (14) pp.1570-1582.

[18] Aharon, M., Elad, M., and Bruckstein, A.M.: The K-SVD: An algorithm for designing over complete dictionaries for sparse representation, *IEEE Trans. Signal Processing*, 2006; (54) pp.4311-4322, November.

[19] Puschel M., Rotteler M.: Algebraic Signal Processing Theory: 2-D hexagonal spatial lattice, IEEE Transactions on Image Processing, 2007; 16(6), pp.1506-1521.

[20] Puschel M., Mura J. F.: Algebraic Signal Processing Theory: 1-D Space, IEEE Transactions on Signal Processing, 2008; 56(8), pp.3586-3599.

[21] Zhihai Z., Ning Z. and Meng L.: Signal Decimation Representation Associate with the Algebraic Signal Processing, *Journal of Computers*, 2018; Vol. 29 No. 2, pp.96-103.