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Implicit K-step Adams-type second - derivative block hybrid methods for the solution of stiff initial value problems in ordinary differential equations

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ABSTRACT

In this paper, we developed a new continuous K-step Adams-Type second-derivative block hybrid methods, using the approach of collocation of the differential system and interpolation of the Taylors series approximate solution at some selected points to get a continuous linear multistep method, which was evaluated at two off-grid points to generate the continuous hybrid linear multistep methods which were evaluated at non-interpolated step points to give CBM's. The K-step methods were augmented by introducing two off-step points to circumvent the Dahlquist zero barrier and upgrade the order of consistency of the methods. Hence, the basic properties of the methods were investigated and found to be consistent, zero-stable, and convergent. The new methods were tested on stiff and highly stiff equations, the results were found to compete favorably with the existing methods in terms of accuracy and error bound.

KEYWORDS

K-step Adams-Type; second-derivative block hybrid methods; collocation; interpolation; and stiff and highly stiff equations; non-interpolated

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(1) INTRODUCTION

Consider the first order initial value problem of the form

$$y' = f(x, y),\ y(x_0) = y_0,\ x \in [a, b]. \tag{1}$$

The solution is in the range $a \le x \le b$, where a and b are finite and we assumed that f satisfies Lipschitz condition, which guarantees the existence and uniqueness of solution of the problem (1). The discrete solution of (1), by linear multistep methods has being studied by many Authors like Lambert [24,25] and for the continuous solution of (1) see Jackson [23], Lie and Norsett [26] and Onumanyi et al [27,28,29]. One important advantage of the continuous lmm over the discrete approach is the ability to provide discrete scheme for simultaneous integration. These discrete schemes can as well be reformulated as general linear method (GLM) Butcher [12,13,14].

Many researchers have worked on the development of continuous linear multistep method in finding solution to (1). These scholars proposed methods with different basis functions among them are [1,2,3, 4, 5,6,7,8,9,11,19,20,21], to mention few, these block methods are self-starting and can directly be applied to stiff problem. In this paper, we present the k-step Adams-type second-derivative Block method with multiple off-step points. The derived schemes will be applied in a block form.

(2) A GENERAL APPROACH TO THE DERIVTION OF THE METHODS

We define the k-step continuous hybrid formula to be of the form

$$y(x)=\sum_{j=1}^{t-1}\alpha_j(x)y_{n+j}+\sum_{j=1}^{s-1}h_n\beta_j(x)f_{n+j},\ n=0,k,2k,\ldots,j \tag{2}$$

Where t *and* s denote the number of interpolation and collocation points respectively and h_n the variable step-size which is valid in the k-step $x_n \le x \le x_{n+k}$. Note that;

$$\alpha_i(x)=\sum_{r=1}^{t}C_n\phi_r(x)+\sum_{r=t+1}^{p}C_n\phi_r(x) \tag{3}$$

$$h_n\beta_i(x)=\sum_{r=1}^{t}C_n\phi_r(x)+\sum_{r=t+1}^{p}C_n\phi_r(x) \tag{4}$$

The Polynomial $\phi_1,\phi_2,\ldots,\phi_p$ is given basis and $p=t+s-1$ is the degree of the polynomial interpolation Y and the collocation point C_i, $i=1,2,\ldots,s$ and C_{ij} are element of an inverse matrix C, for the initial value problems given in the form (1). The formulas in equations (2), (3) and (4) are obtained from the multistep collocation following Onumanyi *et al* [27] which was a generalization of Lie *et al* [26]. The expansion of

$$y(x)\approx Y(x)=\alpha_1\phi_1(x)+\alpha_2\phi_2(x),\alpha_3\phi_3(x),\ldots,\alpha_p\phi_p(x),\qquad x_n \le x \le x_{n+k} \tag{5}$$

Starting with (5) and imposing the following conditions

$$\begin{Bmatrix}\alpha_1\phi_1(x_j)+\cdots+\alpha_p\phi_p(x_j)=y_j,\ j=1,\ldots,t\\ \alpha_1\phi'(x_j)+\cdots+\alpha_p\phi'_p(x_j)=f_i,\ i=1,\ldots,s\end{Bmatrix} \tag{6}$$

Putting (6) in matrix form we have

$$CA=I \tag{7}$$

Where I is the identity matrix of appropriate dimension

$$A=\begin{bmatrix}
1 & x_n & x_n^2 & \cdots & x_n^t & \cdots x_n^{t+m-1}\\
1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^3 & \cdots x_{n+1}^{t+m-1}\\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\
1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^t & \cdots x_{n+t-1}^{t+m-1}\\
0 & 1 & 2\bar{x}_0 & \cdots & t\bar{x}_1^{(t-1)} & \cdots (t+m-1)\bar{x}_1^{t+m-2}\\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\
0 & 1 & 2\bar{x}_m & \cdots & t\bar{x}_m^{(t-1)} & \cdots (t+m-1)\bar{x}_n^{t+m-2}\\
0 & 0 & 2 & \cdots 6\bar{x}_0 & \cdots & (t+m-2)\bar{x}_1^{t+m-3}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots \quad \vdots\\
0 & 0 & 2 & \cdots 6\bar{x}_m & \cdots & (t+m-2)\bar{x}_m^{t+m-3}
\end{bmatrix} \tag{8}$$

and

$$C=\begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,t} & C_{1,t+1} & \cdots & C_{1,t+s} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,t} & C_{2,t+1} & \cdots & C_{2,t+s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ C_{t,1} & C_{t,2} & \cdots & C_{t,t} & C_{t,t+1} & \cdots & C_{t,t+s} \\ C_{t+1,1} & C_{t+1,2} & \cdots & C_{t+1,t} & C_{t+1,t+1} & \cdots & C_{t+1,t+s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ C_{t+s,1} & C_{t+s,2} & \cdots & C_{t+s,t} & C_{t+s,t+1} & \cdots & C_{t+s,t+s} \\ C_{t+s,2} & C_{t+s,3} & \cdots & C_{t+s,t+m} & C_{t+s,t+2} & \cdots & C_{t+s,t+s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ C_{t+s,2} & C_{t+s,3} & \cdots & C_{t+s,t+m} & C_{t+s,t+2} & \cdots & C_{t+s,t+s} \end{bmatrix} \qquad (9)$$

We call A the multi-step collocation and interpolation matrix which has a very simple structure and of dimension $(t+m)x(t+m)$. As can be seen the entries of C are the constants coefficients of the polynomial given in (2) which are to be determined.

(3) SPECIFICATION OF THE METHODS

Two-step method with two-off step points

$$A=\begin{bmatrix} 1 & x_{n+u} & x_{n+u}^2 & x_{n+u}^3 & x_{n+u}^4 & \cdots x_{n+u}^N \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & \cdots Nx_n^{N-1} \\ 0 & 1 & 2x_{n+r} & 3x_{n+r}^2 & 4x_{n+r}^3 & \cdots Nx_{n+r}^{N-1} \\ 0 & 1 & 2x_{n+s} & 3x_{n+s}^2 & 4x_{n+s}^3 & \cdots Nx_{n+s}^{N-1} \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & \cdots Nx_{n+1}^{N-1} \\ \vdots & \vdots & \vdots & \therefore & \therefore & \cdots \vdots \\ 0 & 1 & 2x_{n+k} & 3x_{n+k}^2 & & \cdots x_{n+k}^{N-1} \\ 0 & 0 & 2 & 6x_{n+k-1} & & \cdots N(N-1)x_{n+k-1}^{N-2} \\ 0 & 0 & 2 & 6x_{n+k} & & \cdots N(N-1)x_{n+k}^{N-2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_5 \\ a_{N-1} \\ a_N \end{bmatrix} = \begin{bmatrix} y_{n+u} \\ f_n \\ f_{n+r} \\ f_{n+s} \\ f_{n+1} \\ \vdots \\ f_{n+k} \\ g_{n+k-1} \\ g_{n+k} \end{bmatrix} \qquad (10)$$

The parameters required for equation (10) are k=2 with two-off-grid points

Case I: k=2, with $\left\{\frac{1}{3},\frac{2}{3}\right\}$. Using (10) interpolating and collocating lead to the system of equations writhen in form $CA=I$. yields equation (11).

$$A=\begin{bmatrix}1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+\frac{1}{3}} & 3x^2_{n+\frac{1}{3}} & 4x^3_{n+\frac{1}{3}} & 5x^4_{n+\frac{1}{3}} & 6x^5_{n+\frac{1}{3}} & 7x^6_{n+\frac{1}{3}} \\ 0 & 1 & 2x_{n+\frac{2}{3}} & 3x^2_{n+\frac{2}{3}} & 4x^3_{n+\frac{2}{3}} & 5x^4_{n+\frac{2}{3}} & 6x^5_{n+\frac{2}{3}} & 7x^6_{n+\frac{2}{3}} \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5\end{bmatrix} \tag{11}$$

Similarly, we invert the matrix A in equation (1) above, which leads to the following continuous scheme

$$y(x)=\alpha_1 y_{n+1}+h\left[\beta_0(x)f_n+\beta_{\frac{1}{3}}(x)f_{n+\frac{1}{3}}+\beta_{\frac{2}{3}}(x)f_{n+\frac{2}{3}}+\beta_{n+1}(x)f_{n+1}+\beta_2(x)f_{n+2}\right] \tag{12}$$
$$+h^2\left[g_2(x)f_{n+2}\right]$$

where

$$\alpha_0=1$$

$$\beta_{\frac{1}{3}}=\frac{-3483}{7000}h+\frac{243}{25}\frac{(-x_n+x)^2}{h}-\frac{729}{25}\frac{(-x_n+x)^3}{h^2}+\frac{7533}{200}\frac{(-x_n+x)^4}{h^3}-\frac{(-x_n+x)^5}{h^4}$$
$$+\frac{81}{10}\frac{(-x_n+x)^6}{h^5}-\frac{729}{700}\frac{(-x_n+x)^7}{h^6}$$

$$\beta_{\frac{2}{3}}=-\frac{81}{896}h-\frac{243}{16}\frac{(-x_n+x)^2}{h}+\frac{243}{4}\frac{(-x_n+x)^3}{h^2}-\frac{11907}{128}\frac{(-x_n+x)^4}{h^3}+\frac{2187}{32}\frac{(-x_n+x)^5}{h^4}$$
$$-\frac{1539}{64}\frac{(-x_n+x)^6}{h^5}+\frac{729}{224}\frac{(-x_n+x)^7}{h^6}$$

$$\beta_1=-\frac{37}{120}h+\frac{9(-x_n+x)^2}{h}-\frac{113}{3}\frac{(-x_n+x)^3}{h^2}+\frac{487}{8}\frac{(-x_n+x)^4}{h^3}-\frac{923}{20}\frac{(-x_n+x)^5}{h^4}$$
$$+\frac{33}{2}\frac{(-x_n+x)^6}{h^5}-\frac{9}{4}\frac{(-x_n+x)^7}{h^6}$$

$$\beta_2=-\frac{163}{48000}h+\frac{87}{400}\frac{(-x_n+x)^2}{h}-\frac{151}{150}\frac{(-x_n+x)^3}{h^2}+\frac{6047}{3200}\frac{(-x_n+x)^4}{h^3}-\frac{6931}{4000}\frac{(-x_n+x)^5}{h^4}$$
$$+\frac{243}{320}\frac{(-x_n+x)^6}{h^5}-\frac{99}{800}\frac{(-x_n+x)^7}{h^6}$$

$$\gamma_1 = \frac{17}{420}h^2 - 2(-x_n + x)^2 + \frac{26}{3}\frac{(-x_n+x)^3}{h} - \frac{59}{4}\frac{(-x_n+x)^4}{h^2} + \frac{119}{10}\frac{(-x_n+x)^5}{h^3} - \frac{9}{2}\frac{(-x_n+x)^6}{h^4} + \frac{9}{14}\frac{(-x_n+x)^7}{h^5}$$

$$\gamma_2 = \frac{13}{16800}h^2 - \frac{1}{20}(-x_n + x)^2 + \frac{7}{30}\frac{(-x_n+x)^3}{h} - \frac{71}{160}\frac{(-x_n+x)^4}{h^2} + \frac{83}{200}\frac{(-x_n+x)^5}{h^3} - \frac{3}{16}\frac{(-x_n+x)^6}{h^4} + \frac{9}{280}\frac{(-x_n+x)^7}{h^5}$$

Evaluating the continuous scheme (12) at $x = x_{n+\frac{1}{3}}, x = x_{n+\frac{2}{3}}, x = x_{n+1}, x = x_{n+2}$. We obtain respectively the Adams-Type-Block Hybrid Method which is zero-stable and $A(\alpha) - stable$ by the analysis in the section below. Therefore, the discrete hybrid block method (13).

$$y_n = y_{n+1} - \frac{h}{336000}\left(33700 f_n - 167184 f_{n+\frac{1}{3}} - 30375 f_{n+\frac{2}{3}} - 103600 f_{n+1} - 1141 f_{n+2}\right) + \frac{h^2}{16800}(680 g_{n+1} + 13 g_{n+2})$$

$$y_{n+\frac{1}{3}} = y_{n+1} + \frac{h}{1701000}\left(5700 f_n - 210816 f_{n+1\frac{1}{3}} - 725625 f_{n+\frac{2}{3}} - 204400 f_{n+1} + 1141 f_{n+2}\right) + \frac{h^2}{85050}(20 g_{n+1} - 13 g_{n+2})$$

$$y_{n+\frac{2}{3}} = y_{n+1} - \frac{h}{27216000}\begin{pmatrix} 20900 f_n + 233712 f_{n+\frac{1}{3}} - 3702375 f_{n+\frac{2}{3}} \\ -5569200 f_{n+1} - 13237 f_{n+2} \end{pmatrix} + \frac{h^2}{1360800}(21160 g_{n+1} + 141 g_{n+2})$$

$$y_{n+2} = y_{n+1} + \frac{h}{336000}\begin{pmatrix} 12700 f_n - 104976 f_{n+\frac{1}{3}} + 455625 f_{n+\frac{2}{3}} \\ -148400 f_{n+1} + 121051 f_{n+2} \end{pmatrix} + \frac{h^2}{16800}(7720 g_{n+1} - 643 g_{n+2})$$

(13)

Case II: k=3 with $\left\{\frac{1}{3}, \frac{2}{3}\right\}$. Using (10) interpolating and collocating leads to the system of equation written in the form of (7), yield (14).

$$A = \begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 \\ 0 & 1 & 2x_{n+\frac{1}{3}} & 3x_{n+\frac{1}{3}}^2 & 4x_{n+\frac{1}{3}}^3 & 5x_{n+\frac{1}{3}}^4 & 6x_{n+\frac{1}{3}}^5 & 7x_{n+\frac{1}{3}}^6 & 8x_{n+\frac{1}{3}}^7 \\ 0 & 1 & 2x_{n+\frac{2}{3}} & 3x_{n+\frac{2}{3}}^2 & 4x_{n+\frac{2}{3}}^3 & 5x_{n+\frac{2}{3}}^4 & 6x_{n+\frac{2}{3}}^5 & 7x_{n+\frac{2}{3}}^6 & 8x_{n+\frac{2}{3}}^7 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 & 8x_{n+1}^7 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 & 8x_{n+2}^7 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 & 8x_{n+3}^7 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 \end{bmatrix} \quad (14)$$

Inverting the matrix, A in equation (14) above, which leads to the following continuous schemes;

$$y(x)=\alpha_1 y_{n+1}+h\left[\beta_0(x)f_n+\beta_{\frac{1}{3}}(x)f_{n+\frac{1}{3}}+\beta_{\frac{2}{3}}(x)f_{n+\frac{2}{3}}+\beta_{n+1}(x)f_{n+1}+\beta_2(x)f_{n+2}+\beta_3(x)f_{n+3}\right] \quad (15)$$
$$+h^2\left[g_2(x)f_{n+2}+g_3(x)f_{n+3}\right]$$

where

$$\alpha_0=1$$

$$\beta_0=-\frac{6289}{60480}h-x_n+x-\frac{43}{12}\frac{(-x_n+x)^2}{h}+\frac{691}{108}\frac{(-x_n+x)^3}{h^2}-\frac{1831}{288}\frac{(-x_n+x)^4}{h^3}+\frac{659}{180}\frac{(-x_n+x)^5}{h^4}$$
$$-\frac{131}{108}\frac{(-x_n+x)^6}{h^5}+\frac{3}{14}\frac{(-x_n+x)^7}{h^6}-\frac{1}{64}\frac{(-x_n+x)^8}{h^7}$$

$$\beta_{\frac{1}{3}}=-\frac{11907}{25600}h+\frac{6561}{800}\frac{(-x_n+x)^2}{h}-\frac{729}{32}\frac{(-x_n+x)^3}{h^2}+\frac{175689}{6400}\frac{(-x_n+x)^4}{h^3}-\frac{56133}{3200}\frac{(-x_n+x)^5}{h^4}$$
$$+\frac{39609}{6400}\frac{(-x_n+x)^6}{h^5}-\frac{729}{640}\frac{(-x_n+x)^7}{h^6}+\frac{2187}{25600}\frac{(-x_n+x)^8}{h^7}$$

$$\beta_{\frac{2}{3}}=-\frac{101817}{439040}h-\frac{6561}{784}\frac{(-x_n+x)^2}{h}+\frac{12393}{392}\frac{(-x_n+x)^3}{h^2}-\frac{40095}{896}\frac{(-x_n+x)^4}{h^3}+\frac{123201}{3920}\frac{(-x_n+x)^5}{h^4}$$
$$-\frac{4617}{392}\frac{(-x_n+x)^6}{h^5}+\frac{12393}{5488}\frac{(-x_n+x)^7}{h^6}$$

$$\beta_1=-\frac{1441}{6720}h+\frac{9}{2}\frac{(-x_n+x)^2}{h}-\frac{39}{2}\frac{(-x_n+x)^3}{h^2}+\frac{469}{16}\frac{(-x_n+x)^4}{h^3}-\frac{893}{40}\frac{(-x_n+x)^5}{h^4}+\frac{425}{48}\frac{(-x_n+x)^6}{h^5}$$
$$-\frac{99}{56}\frac{(-x_n+x)^7}{h^6}+\frac{9}{64}\frac{(-x_n+x)^8}{h^7}$$

$$\beta_2=\frac{1723}{134400}h-\frac{243}{400}\frac{(-x_n+x)^2}{h}+\frac{21}{8}\frac{(-x_n+x)^3}{h^2}-\frac{14307}{3200}\frac{(-x_n+x)^4}{h^3}+\frac{1471}{400}\frac{(-x_n+x)^5}{h^4}$$
$$-\frac{911}{600}\frac{(-x_n+x)^6}{h^5}+\frac{171}{560}\frac{(-x_n+x)^7}{h^6}-\frac{153}{6400}\frac{(-x_n+x)^8}{h^7}$$

$$\beta_3=\frac{124459}{47416320}h-\frac{667}{4704}\frac{(-x_n+x)^2}{h}+\frac{27235}{42336}\frac{(-x_n+x)^3}{h^2}-\frac{19135}{16128}\frac{(-x_n+x)^4}{h^3}+\frac{310217}{282240}\frac{(-x_n+x)^5}{h^4}$$
$$-\frac{180763}{338688}\frac{(-x_n+x)^6}{h^5}+\frac{5667}{43804}\frac{(-x_n+x)^7}{h^6}-\frac{611}{50176}\frac{(-x_n+x)^8}{h^7}$$

$$\gamma_2=-\frac{59}{6720}h^2+\frac{9}{20}(-x_n+x)^2-\frac{2(-x_n+x)^3}{h}+\frac{571}{160}\frac{(-x_n+x)^4}{h^2}-\frac{63}{20}\frac{(-x_n+x)^5}{h^3}+\frac{43}{30}\frac{(-x_n+x)^6}{h^4}$$
$$-\frac{9}{28}\frac{(-x_n+x)^7}{h^5}+\frac{9}{320}\frac{(-x_n+x)^8}{h^6}$$

$$\gamma_3 = -\frac{37}{56448}h^2 + \frac{1}{28}(-x_n + x)^2 - \frac{41}{252}\frac{(-x_n + x)^3}{h} + \frac{29}{96}\frac{(-x_n + x)^4}{h^2} - \frac{95}{336}\frac{(-x_n + x)^5}{h^3} \tag{16}$$
$$+ \frac{281}{2016}\frac{(-x_n + x)^6}{h^4} - \frac{27}{784}\frac{(-x_n + x)^7}{h^5} + \frac{3}{896}\frac{(-x_n + x)^8}{h^6}$$

Evaluating the continuous scheme (16) at $x = x_{n+\frac{1}{3}}, x = x_{n+\frac{2}{3}}, x = x_{n+1}, x = x_{n+2}, x = x_{n+3}$, we obtain respectively the Adam's-Type Block Hybrid Method which is zero-stable and $A(\alpha) - stable$ by the analysis in method (13). Therefore, the discrete hybrid block method (17).

$$y_n = y_{n+1} - \frac{h}{237081600}\begin{pmatrix} 24652880f_n - 110270727f_{n+\frac{1}{3}} - 54981180f_{n+\frac{2}{3}} \\ -5083480f_{n+1} + 3039372f_{n+2} + 622295f_{n+3} \end{pmatrix} - \frac{h^2}{282240}(2478g_{n+2} - 185g_{n+3})$$
$$y_{n+\frac{1}{3}} = y_{n+1} + \frac{h}{1080203040}\begin{pmatrix} 3432352f_n - 132774957f_{n+\frac{1}{3}} - 463521528f_{n+\frac{2}{3}} \\ -127734768f_{n+1} + 575064f_{n+2} - 111523f_{n+3} \end{pmatrix} + \frac{h^2}{1285956}(84g_{n+2} + 37g_{n+3})$$
$$y_{n+\frac{2}{3}} = y_{n+1} - \frac{h}{172832486400}\begin{pmatrix} 419506640f_n + 386877137f_{n+\frac{1}{3}} - 33375413340f_{n+\frac{2}{3}} \\ -28754364240f_{n+1} + 928405548f_{n+2} + 141572735f_{n+3} \end{pmatrix} - \frac{h^2}{205752960}(643902g_{n+2} - 41105g_{n+3})$$
$$y_{n+2} = y_{n+1} - \frac{h}{47416320}\begin{pmatrix} 941584f_n + 6751269f_{n+\frac{1}{3}} - 20234124f_{n+\frac{2}{3}} \\ +36345456f_{n+1} + 24666012f_{n+2} + 829291f_{n+3} \end{pmatrix} - \frac{h^2}{56448}(6678g_{n+2} - 229g_{n+3})$$
$$y_{n+3} = y_{n+1} - \frac{h}{7408800}\begin{pmatrix} 337120f_n + 2250423f_{n+\frac{1}{3}} - 6036120f_{n+\frac{2}{3}} \\ +8502480f_{n+1} + 7475832f_{n+2} + 2962105f_{n+3} \end{pmatrix} + \frac{h^2}{8820}(1092g_{n+2} - 415g_{n+3})$$
(17)

(4) ANALYSIS OF THE NEW METHOS

(4.1) In this section, we determine the convergence, construct the regions of absolute stability and obtain the orders and error constants of the new K-step Adams Type Second-derivative block hybrid methods.

Case I: Two step with 2 off- step points.

(4.1.1) Order and Error Constant of the Method (11). Using the method in Chollom, *et al* [15,16]and Fatunla *et al* [19,20], the new K-step Adams Type Second-derivative block hybrid methods has order and error constants as shown below; Let (11) be express in the form

$$A^0 y_{m+1} = \sum_{i=1}^{k} A^i y_{m+1} + h\sum_{j=0}^{k} B^i f_{m-1} \tag{18a}$$

Where h being a fixed mesh size within a block $A^i, B^i, i = 0(1)k$ *are rxr identitymatrix while* Y_m , Y_{m-1} *and* F_{m-1} are vectors of numerical estimates.

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_n\\y_{n+\frac{1}{3}}\\y_{n+\frac{2}{3}}\\y_{n+2}\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}[y_n]+\begin{bmatrix}\frac{-337h}{3360}\\\frac{19h}{5670}\\\frac{-209h}{272160}\\\frac{-127h}{3360}\end{bmatrix}[f_n]+\begin{bmatrix}\frac{-3483h}{7000}&\frac{-81h}{896}&\frac{-37h}{120}&\frac{-63h}{48000}\\\frac{-976h}{7875}&\frac{-215h}{504}&\frac{-14h}{1215}&\frac{163h}{243000}\\\frac{541h}{63000}&\frac{-109h}{8064}&\frac{-221h}{1080}&\frac{-1891h}{3888000}\\\frac{-2187h}{7000}&\frac{1215h}{896}&\frac{-53h}{120}&\frac{17293h}{48000}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{3}}\\f_{n+\frac{2}{3}}\\f_{n+1}\\f_{n+2}\end{bmatrix}$$

$$+\begin{bmatrix}\frac{17h^2}{420}&\frac{13h^2}{16800}\\\frac{2h^2}{8505}&\frac{-13h^2}{85050}\\\frac{529h^2}{34020}&\frac{47h^2}{453600}\\\frac{193h^2}{420}&\frac{-643h^2}{16800}\end{bmatrix}\begin{bmatrix}g_{n+1}\\g_{n+2}\end{bmatrix} \tag{18b}$$

Expanding (18) in Taylor series about x_n and comparing the coefficient of h, gives

$C_0 = C_1 = C_2 = C_3 = C_4 = \cdots = C_7 = 0$ has order $C_p = 08$ in which its order is $p = [7,7,7,7]^T$.

Therefore, the method (11) is of order seven (07) with error constant

$$C_8 = \left[\frac{-1}{777600}, \frac{29}{124002900}, \frac{431}{3968092800}, \frac{19}{1814400}\right]^T.$$

(4.1.2) Consistency: According to (Lambert, 1991)

The block method (11) is consistent since it has order $p = 8 \geq 1$.

(4.1.3) Zero Stability

Definition 4.1: (Lamberts,1991)

A block method is said to be zero-stable if as $h \to 0$, the root $z_i, i = 1(1)k$ of the first characteristic polynomial $\rho(z) = 0$ that is $\rho(z) = \det\left[\sum A^{(i)} Z^{(k-i)}\right] = 0$ satisfies $|Z_i| \leq 1$ and for those roots with $|Z_i| = 1$, the multiplicity must not exceed two. Thus, (11) is expressed in the form

$$\rho(z) = \left| z\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\right| = z^2(z-1)^2 \tag{19}$$

$$\rho(z) = z^2(z-1)^2 = 0, \quad z = 0, 0, 1, 1$$

Hence by definition (4.1), our new method (11) is zero-stable.

(4.1.4) Convergence: The block method (11) is convergent by consistence of Dahlquist theorem below.
Theorem 4.1 (Dahlquist, (1963)

The necessary and sufficient conditions that a continuous linear multistep method be convergent are that it be consistent and zero-stable. Hence our new method (11) is convergent.

(4.1.5) Region of Absolute Stability of method (11)
The stability polynomial for our method (11) is given by

$$h(\omega)=-\frac{1}{5670}\omega^4h^6+\left(\frac{18803}{7654500}\omega^4+\frac{1}{2835}\omega^3\right)h^5+\left(-\frac{31861}{36741600}\omega^4+\frac{731341}{57408750}\omega^3\right)h^4$$
$$+\left(\frac{8069161}{244944000}\omega^3-\frac{1897799}{24494400}\omega^4\right)h^3+\left(\frac{21743711}{122472000}\omega^4+\frac{311966759}{2449440000}\omega^3\right)h^2$$
$$+\left(\frac{3844241}{3024000}\omega^3+\frac{1901359}{3024000}\omega^4\right)h-\omega^4+\omega^3$$

Using MATLAB software, the absolute stability region of the new method (11) is plotted and shown in figure 1.

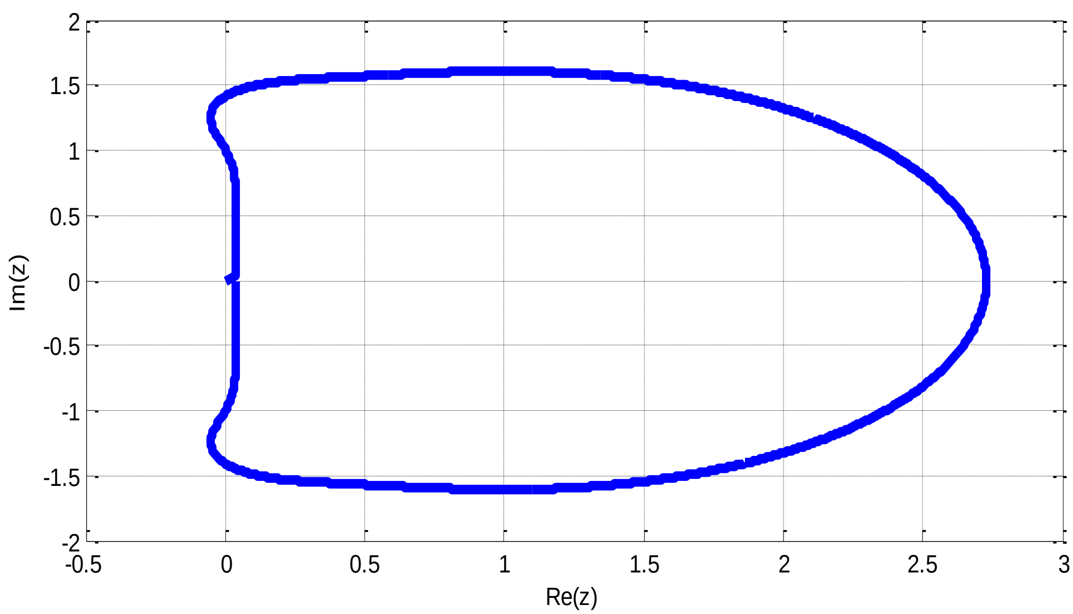

FIGURE 1: Region of absolute Stability of Method (11).

(4.2) Order and error constant of the method (14)

(4.2.1) Order and Error Constant of the Method (14). Using the method in Chollom, et al [15], the new K-step Adams Type Second-derivative block hybrid methods has order and error constants as shown below; Let (14) be express in the form (18a) as shown in (20)

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_n\\y_{n+\frac{1}{3}}\\y_{n+\frac{2}{3}}\\y_{n+2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\left[y_{n+1}\right]+\begin{bmatrix}\frac{-6289h}{60480}\\\frac{2189h}{688905}\\\frac{-107017h}{44089920}\\\frac{-1201h}{60480}\\\frac{-43h}{945}\end{bmatrix}\left[f_n\right]$$

$$+\begin{bmatrix}\frac{-11907h}{25600}&\frac{-101817h}{439040}&\frac{-1441h}{6720}&\frac{1723h}{134400}&\frac{124459h}{47416320}\\\frac{-591h}{480}&\frac{-8831h}{20580}&\frac{-18103h}{153090}&\frac{163h}{306180}&\frac{-111523h}{1080203040}\\\frac{573h}{25600}&\frac{-254347h}{1317120}&\frac{-815033h}{4898850}&\frac{-526307h}{97977600}&\frac{28314547h}{3456649720}\\\frac{729h}{5120}&\frac{-187353h}{439040}&\frac{1717h}{2240}&\frac{4661h}{8960}&\frac{82929h}{47416320}\\\frac{243h}{800}&\frac{-5589h}{6860}&\frac{241h}{210}&\frac{2119h}{2100}&\frac{59242h}{1481760}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{3}}\\f_{n+\frac{2}{3}}\\f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix}+\begin{bmatrix}\frac{-59h^2}{6720}&\frac{-37h^2}{56448}\\\frac{h^2}{15309}&\frac{37h^2}{1285956}\\\frac{1533h^2}{4898880}&\frac{-8221h^2}{41150592}\\\frac{-53h^2}{448}&\frac{229h^2}{56448}\\\frac{13h^2}{105}&\frac{-83h^2}{1764}\end{bmatrix}\begin{bmatrix}g_{n+2}\\g_{n+3}\end{bmatrix} \quad (20)$$

Expanding (20) in Taylor's series about x_n and comparing the coefficient of h, gives

$C_0 = C_1 = C_2 = C_3 = C_4 = \cdots = C_8 = 0$ has order $C_p = 09$ in which its order is $p = [8, 8, 8, 8, 8]^T$ ..

Therefore, the method (14) is of order eight (08) with error constant

$$C_9 = \left[\frac{1}{595350}, \frac{-4103}{3124873080}, \frac{12491}{31248730800}, \frac{1}{176400}, \frac{89}{4762800}\right]^T.$$

(4.2.2) Consistency

The hybrid block method (14) is said to be consistent, since it is of order nine (09).

(4.2.3) Zero- Stability

On the application of definition (4.1), on the method (14), we obtain

$$\rho(z) = \left| z\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\right| = z^3(z-1)^2$$

$$\rho(z) = z^3(z-1)^2 = 0, \quad z = 0, 0,0, 1, 1$$

Hence, our method (14) is zero-stable

(4.2.4) Convergence:
On the application of theorem (4.1), it is clear that method (14) is convergent since it is consistent and zero-stable.

(4.2.5) Region of Absolute Stability:
The stability polynomial of the method (14) is given by

$$h(\omega)=\frac{1}{5040}\omega^5h^7+\left(\frac{22406}{12301875}\omega^4-\frac{3091889}{12002256000}\omega^5\right)h^6+\left(\frac{12813390548551}{459356342760000}\omega^4-\frac{2366851601}{288054144000}\omega^5\right)h^5$$
$$+\left(\frac{9126835206827}{57419542845000}\omega^4+\frac{3315915103}{72013536000}\omega^5\right)h^4+\left(\frac{5359685255059301}{9799601978880000}\omega^4-\frac{46826296879}{11522165760000}\omega^5\right)h^3$$
$$+\left(\frac{3424693729511}{2304433152000}\omega^4-\frac{870107482487}{2304433152000}\omega^5\right)h^2+\left(\frac{1708595177}{960180480}\omega^4+\frac{1171946263}{960180480}\omega^5\right)h-\omega^5+\omega^4$$

Using MATLAB Software, the absolute stability1 region of the new method (14) is plotted and shown in figure 2.

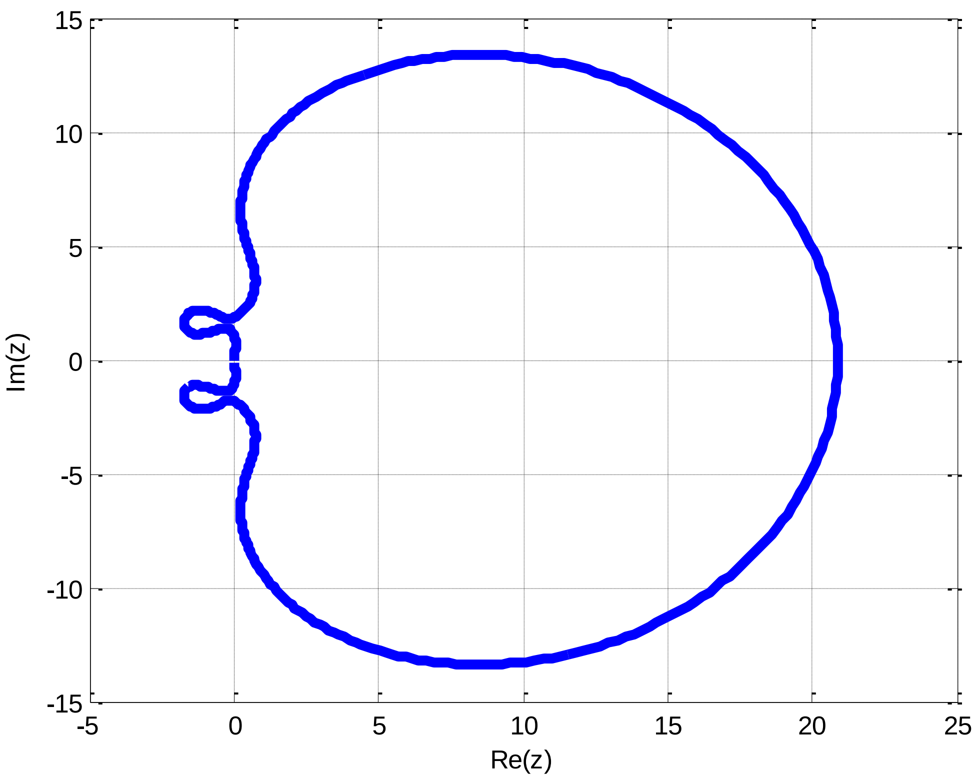

FIGURE 2: Region of absolute Stability of method (14)

(5) NUMERICAL EXPERIMENTS
To check the behavior of the Newly constructed k-Step Adams-Type second-derivative Block Hybrid Methods (11) and (14) are tested on stiff differential equations, we solve well known numerical problems using a fixed step size. The test problems are solved and results obtained are compared with the existing once in the literature to illustrate their potentials.

Example 1:

Consider the stiff problem $y'(x) = 5y(x), \quad y(0) = 1, \quad h = \frac{1}{100}, \quad$ *whose exact solution is* $y(x) = \exp(5x)$.

Source: Yakusak and Adeniyi (2015)

Example 2:

Consider the highly stiff problem

$$y' + 4y = 20, \ y(0) = -4, \ y(0) = 2, \ 0 \le x \le 1, \ h = \frac{1}{100}$$

with the exact solution

$$y(x) = 5 - 3e^{-4x}$$

Source: Badmus, Yahaya and Subair (2014)

Example 3:

We consider the highly stiff problem of ordinary differential equation

$$y' = -\alpha(y_0 - F(x)) + F'(x), \ y(x_0) = y_0, \ x \in [0, 0, 1], \ h = 0.01$$

with the exact solution

$$y(x) = y_0 + \exp(-\alpha x) - 1, \ \text{where } \alpha = 10, \ F(x) = 0, \ x_0 = 0 \text{ and } y_0 = 1$$

Source: Rufai M.A., Duromola, M. K., and Ganiyu, A. A. (2016).

TABLE 1: Comparing the absolute errors in the new methods with Yakusak and Adeniyi (2015) for Example 1

h=0.01 X	Exact Solution of New Method (11)	Exact Solution of New Method 14)	Absolute Error in method (11)	Absolute Error in method (14)	Absolute Error in Yakusak & Adeniyi (2015)
0.01	1.0512710963764	1.05127109637	-5.52000E-17	3.7000E-18	0.0000000000
0.02	1.1051709180756	1.10517091807	0.000000000	-8.3000E-18	0.0000000000
0.03	1.1618342427283	1.16183424272	-5.83600E-16	-3.6800E-17	0.0000000000
0.04	1.2214027581601	1.22140275816	-1.09880E-15	-3.4300E-17	0.0000000000
0.05	1.2840254166877	1.28402541668	-1.22260E-15	-5.0300E-17	0.0000000000
0.06	1.3498588075760	1.34985880757	-1.82150E-15	-8.5500E-17	1.000000E-10
0.07	1.4190675485932	1.41906754859	-1.98940E-15	-8.4900E-17	1.000000E-10
0.08	1.4918246976412	1.49182469764	-2.68410E-15	-1.0560E-16	1.000000E-10
0.09	1.5683121854901	1.56831218549	-2.90410E-15	-1.4890E-16	1.000000E-10
0.10	1.6487212707000	1.64872127070	-3.70790E-15	-1.5080E-16	1.000000E-10

TABLE 2: Comparing the absolute errors in the new methods with Badmus, Yahaya and Subair (2014) for Example 2

h=0.01 X	Exact Solution of New Method (11)	Exact Solution of New Method 14)	Absolute Error in method (11)	Absolute Error in method (14)	Absolute Error in Badmus,Yahaya & Subair (2014)
0.01	2.117631682543	2.117631682543	2.35000E-17	1.20000E-18	2.00000E-14
0.02	2.230650960840	2.230650960840	2.17000E-16	3.10000E-18	3.00000E-14
0.03	2.339238689848	2.339238689848	2.30200E-16	1.23000E-17	8.00000E-14
0.04	2.443568633101	2.443568633101	4.00800E-16	1.07000E-17	1.20000E-12
0.05	2.543807740766	2.543807740766	4.05100E-16	1.40000E-17	1.00000E-12
0.06	2.640116416800	2.640116416800	5.55000E-16	2.18000E-17	2.69700E-11
0.07	2.732648775632	2.732648775633	5.51600E-16	2.02000E-17	5.58000E-12
0.08	2.821552888779	2.821552888779	6.83100E-16	2.27000E-17	6.21400E-10
0.09	2.906971021787	2.906971021787	6.73200E-16	2.93000E-17	2.70480E-10
0.10	2.989039861893	2.989039861893	7.88000E-16	2.73000E-17	1.45710E-08

TABLE 3: Comparing the absolute errors in the new methods with Rufai *et.al.* (2016) for Example 3

h=0.01 X	Exact Solution of New Method (11)	Exact Solution of New Method 14)	Absolute Error in method (11)	Absolute Error in method (14)	Absolute Error in Rufai et.tal. (2016)
0.01	0.904837418035	0.904837418035	1.064197E-14	1.335650E-15	1.079154E-12
0.02	0.818730753078	0.818730753078	1.006416E-13	3.421540E-15	1.952918E-12
0.03	0.740818220682	0.740818220682	9.977720E-14	1.384129E-14	2.650610E-12
0.04	0.670320046035	0.670320046035	1.647967E-13	1.153464E-14	3.197828E-12
0.05	0.606530659712	0.606530659713	1.562478E-13	1.386704E-14	3.616893E-12
0.06	0.548811636093	0.548811636094	2.023862E-13	2.050776E-14	3.927240E-12
0.07	0.496585303791	0.496585303791	1.889671E-13	1.782317E-14	4.145766E-12
0.08	0.449328964117	0.449328964117	2.209331E-13	1.866815E-14	4.287136E-12
0.09	0.406569597403	0.406569597406	2.046903E-13	2.278879E-14	4.364056E-12
0.10	0.367879441171	0.367879441171	2.261059E-13	2.007710E-14	4.387513E-12

CONCLUSION

In this paper a new class of implicit K-step Adams-Type Second-Derivative Block Hybrid Methods (SDBHAMs) is considered for the numerical solution of stiff IVPs in ODEs. The addition of off-grid points allowed the adoption of linear multistep procedure which helps the zero-stability barrier, upgraded the order of accuracy of the new methods and to obtain very low error constants. The new methods where tested on some stiff and highly stiff problems, and compared with some exiting methods cited in the literature, shows that our new methods are superior and performed better than the methods in the literature and are well suited for the integration of stiff equations in ODEs.

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