IRESPUB Journal of Engineering & Computer Sciences

Volume: 1 | Issue: 1 | NOV - DEC 2021 | Available Online: www.irespub.com

Implicit second-derivative runge-kutta collocation methods for the solution of systems of initial value problems

Donald J. Z[1*], G.M Kumleng[2], and Yusuf. S[2]

[1,2]Department of Mathematics, Adamawa State University, Mubi, Nigeria

[2]Department of Mathematics, University of Jos, Plateau State, Nigeria

ABSTRACT

In this paper, we present a class of implicit Second-derivative Runge-Kutta collocation methods designed for the numerical solution of systems of initial value problems that are derived and studied. We also discuss the difficulty associated with large regions of absolute stability. In this case, one must take advantage of the second derivative terms in the methods. We involve the introduction of collocation at the two endpoints of the integration interval in addition to the Gaussian interior collocation points and also the introduction of a different class of basic second derivative methods. With these modifications, fewer function evaluations per step are achieved. The stability and consistency properties of the methods are investigated, with the solution curves of the new methods. Numerical examples are given to illustrate the accuracy and efficiency of the proposed methods.

KEYWORDS

gaussian interior points; block hybrid scheme; continuous scheme; system of equations; second-derivative runge-Kutta methods

CORRESPONDING AUTHOR

Donald J. Z.

INTRODUCTION

In this paper, we present a new class of implicit second-derivative Runge-Kutta (SDRK) collocation methods for the numerical solution of initial value problems for systems of ordinary differential equations (ODEs), of the form

$$\left.\begin{array}{l} y_1 = f_1(x, y_1, y_2, \ldots, y_n) \\ y_2 = f_2(x, y_1, y_2, \ldots, y_n) \\ \vdots \\ y_n = f_n(x, y_1, y_2, \ldots, y_n) \end{array}\right\} \quad (1a)$$

with initial conditions

$$\left.\begin{array}{l} y_1(x_0) = y_1^0 \\ y_2(x_0) = y_2^0 \\ \vdots \\ y_n(x_0) = y_n^0 \end{array}\right\} \quad (1b)$$

On the finite interval $I=[x_0,x_N]$, where $y:[x_0,x_N]\rightarrow\Re^m and\ f:[x_0,x_N]\times\Re^m\rightarrow\Re^m$ is continuous and differentiable. The motivation for studying the implicit second-derivative Runge-Kutta collocation methods, particularly, the Gauss–Runge–Kutta collocation family, is that, collocation at the Gauss points leads to Runge-Kutta methods which are symmetric and algebraically stable, Burrage and Butcher [1979].we therefore, extend the concept adopted in Yakubu [2003, 2010, 2011, 2015, 2016] and Donald, Skwame and Dominic [2015] that the only symmetric algebraically stable collocation methods are those based on Gauss points. The inclusion of the two endpoints of the integration interval as collocation points in addition to the Gaussian interior collocation points make them more advantageous, because this minimizes the number of internal function evaluations necessary to achieve a given order of accuracy. Secondly, a substantial increase in efficiency maybe achieved by the numerical integration methods which utilize the second-derivative terms. Thirdly, the relatively good stability properties enjoyed by these methods make them more efficient for the numerical integration of systems having Jacobians with eigenvalues lying close to the imaginary axis Adesanya, Fotta and Onsachi[2016] and Akinfenwa, Abdulganiy, Akinnukawe ,Okunuga and Rufai [2017].

This paper focused on the derivation of a class of implicit second-derivative Runge-Kutta (SDRK) collocation methods for solving systems of initial value problem of ordinary differential equations. Yakubu, Kumleng and Markus [2017], developed second-derivative Runge-Kutta collocation methods based on Lobatto nodes for solving stiff ODEs and yields a very good results, their work motivates us to derive a class of implicit Second-derivative Runge-Kutta collocation methods designed for the numerical solution of systems of initial value problems.

1. A GENERAL APPROACH TO THE DERIVATION OF THE SDRK COLLOCATION METHODS

In this section, we describe the general derivation of the special class of implicit second-derivative Runge-Kutta collocation methods for direct integration of initial value problems of the form (1). We consider the multistep collocation approach of Onumanyi et al. [1994] and now extend to second derivative of the form,

$$y(x)=\sum_{j=0}^{r-1}\alpha_j(x)y_{n+j}+h\sum_{j=0}^{s-1}\beta_j(x)\bar{y}'_{n+j}+h^2\sum_{j=0}^{t-1}\gamma_j(x)\bar{y}''_{n+j} \tag{2}$$

We set the sum $p=r+s+t$ where, r denotes the number of interpolation points used, and $s>0, t>0$ are distinct collocation points. Here $\alpha_j(x)$, $\beta_j(x)$ and $\gamma(x)$ are parameters of the methods which are to be determined. They are assumed to be polynomials of the form

$$\alpha_j(x)=\sum_{i=0}^{p-1}\alpha_{j,i+1}x^i\ ,\quad h\beta_j(x)=\sum_{i=0}^{p-1}h\beta_{j,i+1}x^i\ ,\quad h^2\gamma_j(x)=\sum_{i=0}^{p-1}h^2\gamma_{j,i+1}x^i \tag{3}$$

We find it convenient to introduce the following polynomials

$$\rho(\xi)=\sum_{i=0}^{p-1}\alpha_i\xi^i\ ,\qquad \sigma(\xi)=\sum_{i=0}^{p-1}\beta_i\xi^i\ ,\qquad \tau(\xi)=\sum_{i=0}^{p-1}\gamma_i\xi^i$$

which we shall call the first, second and third characteristic polynomials respectively of (2).
Here, our aim is to utilize not only the interpolation points $\{x_j\}$ but also several collocation points on the interpolation interval of (2). This means that we employ a special type of Hermite interpolation for $y(x)$. Substituting (3) into (2) we have

$$y(x)=\sum_{j=0}^{r-1}\sum_{i=0}^{p-1}\alpha_{j,i+1}x^i y_{n+j}+h\sum_{j=0}^{s-1}\sum_{i=0}^{p-1}\beta_{j,i+1}x^i\ \bar{y}'_{n+j}{}^{i}+h^2\sum_{j=0}^{t-1}\sum_{i=0}^{p-1}\gamma_{j,i+1}+x^i\ \bar{y}''_{n+j}$$

$$=\sum_{i=0}^{p-1}\left\{\sum_{j=0}^{r-1}\alpha_{j,i+1}y_{n+j}+h\sum_{j=0}^{s-1}\beta_{j,i+1}\bar{y}'_{n+j}+h^2\sum_{j=0}^{t-1}\gamma_{j,i+1}y''_{n+j}\right\}x^i\ . \tag{4}$$

Writing

$$\phi_i=\sum_{i=0}^{p-1}\left\{\sum_{j=0}^{r-1}\alpha_{j,i+1}y_{n+j}+h\sum_{j=0}^{s-1}\beta_{j,i+1}\bar{y}'_{n+j}+h^2\sum_{j=0}^{t-1}\gamma_{j,i+1}y''_{n+j}\right\}$$

Equation (4) reduces to

$$y(x)=\sum_{i=0}^{p-1}\phi_i x^i. \tag{5}$$

Here $\{c_{n+j}\}$ are collocation points distributed on the step-points array, y_{n+j} is the interpolation data of $y(x)$ on x_{n+j}, $\bar{y}'_{n+j}$ and $\bar{y}''_{n+j}$ are the collocation data of $y'(x)$ and $y''(x)$, respectively, on $\{c_{n+j}\}$. We set the sum $r+s+t$ to be equal to p so as to be able to determine $\{\alpha_i\}$ in (2) uniquely.

To fix the parameters $\alpha_i\,(i=0,1,\ldots,p-1)$, we impose the following conditions:

$$\alpha(x_{n+j})=y_{n+j}\ ,\qquad (j=0,1,2,\cdots,r-1) \tag{6}$$

$$\beta'(c_{n+j})=\bar{y}'_{n+j}\qquad (j=0,1,2,\cdots s-1) \tag{7}$$

$$\gamma''(c_{n+j})=\bar{y}''_{n+j}\qquad (j=0,1,2,\cdots t-1) \tag{8}$$

In fact, equations (6) to (8) can be expressed in the matrix-vector form by

$$M\alpha=y \tag{9}$$

where the $p-square\ matrix\ M$, $the\ p-vectors\alpha\ and\ y$ are defined as follows:

$$M=\begin{pmatrix}1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & \cdots & x_n^{p-1}\\ 0 & x_{n+1} & x^2{}_{n+1} & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & \cdots & x_{n+1}^{p-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_{n+s-1} & x_{n+s-1}^2 & x_{n+s-1}^3 & x_{n+s-1}^4 & x_{n+s-1}^5 & \cdots & x_{n+s-1}^{p-1}\\ 0 & 1 & 2c_n & 3cx_n^2 & 4cx_n^3 & 5cx_n^4 & \cdots & D'c_n^{p-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 1 & 2c_{n+s-1} & 3c_{n+s-1}^2 & 4c_{n+s-1}^3 & 5x_{n+s-1}^4 & \cdots & D'c_{n+s-1}^{p-2}\\ 0 & 0 & 2 & 6c_n & 12c_n^2 & 20c_n^3 & \cdots & D''c_n^{p-3}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 2 & 6c_{n+s-1} & 12c_{n+s-1}^2 & 20c_{n+s-1}^3 & \cdots & D''x_{n+s-1}^{p-3}\end{pmatrix} \tag{10}$$

$$\alpha=(\alpha_0,\alpha_1,\alpha_{2,}\cdots,\alpha_{p-1})^T,\ \ y=(y_n,\cdots,y_{n+r-1},\ \bar{y}'_n,\cdots,\bar{y}'_{n+s-1},\ \bar{y}''_n,\cdots,\bar{y}''_{n+s-1})^T$$

where $D' = (p-1)\ and\ D'' = (p-1)(p-2)$ represent first and second derivatives respectively. Similar to the Vandermonde matrix, M in (9) is non-singular. Consequently, equation (9) has the unique solution given by

$$\alpha = M^{-1} y \tag{11}$$

The interpolation polynomial $y(x)$ in (5) can now be expressed explicitly as follows:

$$y(x) = \left\{ \sum_{j=0}^{r-1} \alpha_{j,p-1} y_{n+j} + h \sum_{j=0}^{s-1} \beta_{j,p-1} \bar{y}'_{n+j} + h^2 \sum_{j=0}^{t-1} \gamma_{j,p-1} \bar{y}''_{n+j} \right\} \left(1, x, x^2, \ldots, x^{p-1}\right)^T . \tag{12}$$

Recall that $p+s+t$, such that equation (12) becomes

$$y(x) = \left\{ \sum_{j=0}^{r-1} \alpha_{j,r+s+t-1} y_{n+j} h \sum_{j=0}^{s-1} \beta_{j,r+s+t-1} \bar{y}'_{n+j} + h^2 \sum_{j=0}^{t-1} \gamma_{j,r+s+t-1} y''_{n+j} \right\} \left(1, x, x^2, \ldots, x^{r+s+t-1}\right)^T . \tag{13}$$

Expanding (13) fully, gives the continuous scheme;

$$y(x) = (y_n, \cdots, y_{n+r-1,}\ \ \bar{y}'_n, \cdots, \bar{y}'_{n+s-1},\ \ \bar{y}''_n, \cdots, \bar{y}''_{n+t-1}) U^T \left(1, x, x^2, \cdots, x^{r+s+t-1}\right)^T .$$

where T denotes transpose of the matrix U in (11) and the vector $\left(1, x, x^2, \cdots, x^{r+s+t-1}\right)$.

In the second-derivative methods, we see that, in addition to the computation of the f -values at the internal stages in the standard Runge-Kutta methods Butcher [2014], the modified methods involve computing g-values, where g is defined by Butcher and Hojjati [2005] as

$y''(x) = g(y(x))$, the component number i of $g(y(x))$ can be written as,

$$g_i(y(x)) = \sum \frac{\partial f_i(y(x))}{\partial y_i} f_j(y(x)), \quad i = 1,2,\ldots,m.$$

According to Chan and Tsai [2010] these methods can be practical if the costs of evaluating g are comparable to those in evaluating f and can even be more efficient than the standard Runge-Kutta methods if the number of function evaluations is fewer. It is convenient to rewrite the coefficients of the defining method (13) evaluated at some points in the block matrix form as

$$Y = e \otimes y_n + h(A \otimes I_N) F(Y) + h^2 (\hat{A} \otimes I_N) G(Y), \tag{14}$$

$$y_{n+1} = y_n + h(b^T \otimes I_N) F(Y) + h^2 (\hat{b}^T \otimes I_N) G(Y),$$

where, $A = [a_{ij}]_{sxs}$, $\hat{A} = [\hat{a}_{ij}]_{sxs}$ indicate the dependence of the stages on the derivatives found at the other stages and $b = [b_i]_{sx1}$, $\hat{b} = [\hat{b}_i]_{sx1}$ are vectors of quadrature weights showing how the final result depends on the derivatives computed at the various stages, I is the identity matrix of size equal to the differential equation system to be solved and N is the dimension of the system. Also $\otimes$ is the Kronecker product of two matrices and e is the s×1 vector of units. For simplicity, we write the method in Yakubu [2017] and Yusuf [2019] as follows:

$$Y = y_n + hAF(Y) + h^2 \hat{A} G(Y), \tag{15}$$

$$y_{n+1} = y_n + hb^T F(Y) + h^2 \hat{b}^T G(Y),$$

and the block vectors in $\Re^{sN}$ are defined by

$$Y=\begin{bmatrix}Y_1\\Y_2\\\vdots\\Y_s\end{bmatrix},\qquad F(Y)=\begin{bmatrix}f(Y_1)\\f(Y_2)\\\vdots\\f(Y_s)\end{bmatrix},\qquad G(Y)=\begin{bmatrix}g(Y_1)\\g(Y_2)\\\vdots\\g(Y_s)\end{bmatrix}. \tag{16}$$

where s denotes stage, values used in the computation of the step $Y_1, Y_2, \ldots, Y_S$.

The coefficients of the Implicit Second-Derivative Runge-Kutta methods can be conveniently represented more compactly in an extended partitioned Butcher Tableau, of the form

$$\begin{array}{c|c||c} c & A & \hat{A} \\ \hline & b^T & \hat{b}^T \end{array} \tag{17}$$

where $c=[1]_{sx1}$ is the abscissa vectors which indicates the position within the step of the stage values.

2. THE SPECIFICATION OF METHODS

2.1 Fifth Order Implicit Second-Derivative Runge-Kutta Collocation Method

For the first implicit second-derivative Runge-Kutta collocation method of order five, we define $\xi=(x-x_n)$ and consider the zeros of Legendre polynomial of degree two in the symmetric interval $[-1,1]$, which were transformed into the standard interval $[x_n, x_{n+1}]$. The proposed continuous scheme in (13) can now be written as,

$$y(x)=\alpha_0(x)y_n+h[\beta_1(x)f_{n+u}+\beta_2(x)f_{n+v}]+h^2[\gamma_1(x)g_{n+u}+\gamma_2(x)g_{n+v}] \tag{18}$$

Where

$$\alpha_0(x)=1$$

$$\beta_1(x)=\frac{h\sqrt{2}}{160}\left(40t\sqrt{2}+25t+240t^2-720t^3+640t^4-192t^5\right)$$

$$\beta_2(x)=\frac{h\sqrt{2}}{160}\left(-25t+40t\sqrt{2}-240t^2+720t^3-640t^4+192t^5\right)$$

$$\gamma_1(x)=-\frac{h^2(-5+2\sqrt{2})}{16320}\begin{pmatrix}-1095-540\sqrt{2}+6000t-12240h\,t^2-2720t^2\sqrt{2}+10560t^3\\+960t^3\sqrt{2}-3264t^4+2400t\sqrt{2}\end{pmatrix}t$$

$$\gamma_2(x)=-\frac{h^2(5+2\sqrt{2})}{16320}\begin{pmatrix}1095t-540t\sqrt{2}-6000t+12240t^3-2720t^2\sqrt{2}-10560t^3\\+960t^3\sqrt{2}+3264t^5+2400t\sqrt{2}\end{pmatrix}t$$

$$\gamma_3(x)=\frac{t}{240}\left(15-120th^2+400h^2t^2-480t^3h^2+192t^4h^2\right)$$

Evaluating the continuous scheme $y(x)$in (18) at the points $x = x_{n+1}, x_{n+u}$ *and* x_{n+v} (where u and v are the zeros of Legendre polynomial of degree 2) we obtain the implicit second- derivative Runge-Kutta collocation method of uniformly order 5with only 2-stages with the following block hybrid discrete scheme:

$$y_{n+1} = y_n + \frac{h}{120}\left[(32+14\sqrt{2})f_{n+u} + (32-14\sqrt{2})f_{n+v} + 56f_{n+1}\right] + \frac{h^2}{120}\left[-(7-6\sqrt{2})g_{n+u} - (7+6\sqrt{2})g_{n+v}\right]$$
$$y_{n+u} = y_n + \frac{h}{1920}\left[(128+90\sqrt{2})f_{n+u} + (384-314\sqrt{2})f_{n+v} + (448-256\sqrt{2})f_{n+1}\right] + \frac{h^2}{1920}\left[-(143+80\sqrt{2})g_{n+u} + (1-16\sqrt{2})g_{n+v}\right]$$
$$y_{n+v} = y_n + \frac{h}{1920}\left[(384+314\sqrt{2})f_{n+u} + (128-90\sqrt{2})f_{n+v} + (448+256\sqrt{2})f_{n+1}\right] + \frac{h^2}{1920}\left[(1+16\sqrt{2})g_{n+u} + (143+80\sqrt{2})g_{n+v}\right]$$

(19)

Converting the block hybrid discrete scheme to implicit second-derivative Runge-Kutta method and using (16) we write the method as,

$$y_n = y_{n-1} + h\left(\frac{1}{10} + \frac{7\sqrt{2}}{60}\right)F_1 + h\left(\frac{1}{10} - \frac{7\sqrt{2}}{60}\right)F_2 + h\left(\frac{4}{5}\right)F_3 + h^2\left(\frac{\sqrt{2}}{48}\right)G_1 - h^2\left(\frac{\sqrt{2}}{48}\right)G_2 \qquad (20)$$

where the internal stage values at the n^{th} step are calculated as,

$$Y_1 = y_{n-1},$$

$$Y_2 = y_{n-1} + h\left(\frac{1}{15} + \frac{3\sqrt{2}}{64}\right)F_1 + h\left(\frac{1}{4} - \frac{157\sqrt{2}}{960}\right)F_2 + h\left(\frac{57}{240} - \frac{2\sqrt{2}}{15}\right)F_3 - h^2\left(\frac{143}{1920} + \frac{\sqrt{2}}{4}\right)G_1 + h^2\left(\frac{61}{1920} - \frac{23\sqrt{2}}{960}\right)G_2$$

$$Y_3 = y_{n-1} + h\left(\frac{1}{4} + \frac{157\sqrt{2}}{960}\right)F_1 + h\left(\frac{1}{15} - \frac{3\sqrt{2}}{64}\right)F_2 + h\left(\frac{57}{240} + \frac{2\sqrt{2}}{15}\right)F_3 + h^2\left(\frac{1}{1920} + \frac{\sqrt{2}}{120}\right)G_1 - h^2\left(\frac{143}{1920} + \frac{\sqrt{2}}{4}\right)G_2$$

$$Y_4 = y_{n-1} + h\left(\frac{1}{10} + \frac{7\sqrt{2}}{60}\right)F_1 + h\left(\frac{1}{10} - \frac{7\sqrt{2}}{60}\right)F_2 + h\left(\frac{4}{5}\right)F_3 + h^2\left(\frac{\sqrt{2}}{48}\right)G_1 - h^2\left(\frac{\sqrt{2}}{48}\right)G_2$$

where the stage derivatives are calculated as follows:

$$F_1 = f\left(x_{n-1} + h(0), Y_1\right),$$
$$F_2 = f\left(x_{n-1} + h\left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right), Y_2\right),$$
$$F_3 = f\left(x_{n-1} + h\left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right), Y_3\right),$$
$$F_4 = f\left(x_{n-1} + h(1), Y_4\right).$$

The implicit second-derivative Runge-Kutta collocation method has order p = 5. Writing the method in an extended Butcher Tableau (16), we have

$$
\begin{array}{c|ccc|ccc}
\frac{2-\sqrt{2}}{4} & \frac{(128+90\sqrt{2})}{1920} & \frac{(384-314\sqrt{2})}{1920} & \frac{(448-256\sqrt{2})}{1920} & \frac{-143+80\sqrt{2}}{1920} & \frac{61-46\sqrt{2}}{1920} & 0 \\
\frac{2+\sqrt{2}}{4} & \frac{(384+314\sqrt{2})}{1920} & \frac{(128-90\sqrt{2})}{1920} & \frac{(448+256\sqrt{2})}{1920} & \frac{(1+16\sqrt{2})}{1920} & \frac{(143+80\sqrt{2})}{1920} & 0 \\
1 & \frac{(32+14\sqrt{2})}{120} & \frac{(32-14\sqrt{2})}{120} & \frac{56}{120} & \frac{-7-6\sqrt{2}}{120} & \frac{-7+6\sqrt{2}}{120} & 0 \\
\hline
 & \frac{(32+14\sqrt{2})}{120} & \frac{(32-14\sqrt{2})}{120} & \frac{56}{120} & \frac{-7-6\sqrt{2}}{120} & \frac{-7+6\sqrt{2}}{120} & 0
\end{array}
$$

2.2 A Sixth Order Implicit Second-Derivative Runge-Kutta Collocation Method

Next, as the order of the method being sought for increases, the algebraic conditions on the coefficients of the method become increasingly complicated. However, we consider again the two end points of the integration interval as collocation points in addition to the Gaussian interior collocation points, obtained in the same manner as in method (18) with $p_3(x)=0$, Legendre polynomial of degree 3 as follows,

$$
\left.
\begin{aligned}
\bar{x}_0 &= x_{n+u}, \quad u=\left(\frac{1}{2}-\frac{\sqrt{3}}{4}\right) \\
\bar{x}_1 &= x_{n+w}, \quad w=\frac{1}{2} \\
\bar{x}_2 &= x_{n+v}, \quad v=\left(\frac{1}{2}+\frac{\sqrt{3}}{4}\right)
\end{aligned}
\right\}
$$

which are also valid in the interval $[x_n, x_{n+1}]$, expanding (2) to obtain the proposed continuous scheme in (13) takes the following form:

$$y(x)=\alpha_0(x)y_n+h[\beta_1(x)f_{n+u}+\beta_2(x)f_{n+w}+\beta_3(x)f_{n+v}]+h^2[\gamma_1(x)g_{n+u}+\gamma_2(x)g_{n+w}+\gamma_3(x)g_{n+v}] \tag{21}$$

where

$$\alpha_0(x)=1$$

$$\beta_1(x)=-\frac{2h\sqrt{3}}{135}\begin{pmatrix}-10\sqrt{3}-15-20t\sqrt{3}-75t+120t^2\sqrt{3}+500t^2-1050t^3-160t^3\sqrt{3} \\ +64t^4\sqrt{3}+960t^4-320t^5\end{pmatrix}t$$

$$\beta_2(x)=\frac{t\,h}{45}\left(5-80t+480t^2-640t^3+256t^4\right)$$

$$\beta_3(x)=-\frac{2h\sqrt{3}}{135}\begin{pmatrix}-10\sqrt{3}+15-20t\sqrt{3}+75t+120t^2\sqrt{3}-500t^2+1050t^3-160t^3\sqrt{3} \\ +64t^4\sqrt{3}-960t^4+320t^5\end{pmatrix}t$$

$$\gamma_1(x) = -\frac{h\,t}{540}\begin{pmatrix} 30+15\sqrt{3}-150\,t\,h\sqrt{3}-330\,t\,h+1240\,t^2h+420\,t^2h\,\sqrt{3}-2220\,t^3h-480\,t^3h\,\sqrt{3} \\ +192\,t^4h\,\sqrt{3}+1920\,t^4h-640\,t^5h \end{pmatrix}$$

$$\gamma_2(x) = \frac{h\,t}{54}\left(256\,t^5h-768\,t^4h+816\,t^3h-352\,t^2h+51\,t\,h+3\right)$$

$$\gamma_3(x) = \frac{h\,t}{540}\begin{pmatrix} -30+15\sqrt{3}+330\,t\,h-150\,t\,h\,\sqrt{3}+420\,t^2h\,\sqrt{3}-1240\,t^2h-480\,t^3h\,\sqrt{3}+2220\,t^3h \\ +192\,t^4h\,\sqrt{3}-1920\,t^4h+640\,t^5h \end{pmatrix}$$

Evaluating the proposed continuous scheme $y(x)$ in (19) at the points $x = x_{n+1}, x_{n+u}$ *and* x_{n+v} (where u and v are the zeros of Legendre polynomial of degree 3) we obtain the block hybrid discrete scheme as follows:

$$y_{n+1} = y_n + \frac{h}{180}\left[48f_{n+u} + 84f_{n+w} + 48f_{n+v}\right] + \frac{h^2}{180}\left[\sqrt{3}g_{n+u} - \sqrt{3}g_{n+v}\right]$$

$$y_{n+u} = y_n + \frac{h}{34560}\left[\left(4608-1396\sqrt{3}\right)f_{n+u} + \left(8064-4608\sqrt{3}\right)f_{n+w} + \left(4608-2636\sqrt{3}\right)f_{n+v}\right]$$
$$+\frac{h^2}{34560}\left[-\left(271-96\sqrt{3}\right)g_n - 40g_{n+w} + \left(161-96\sqrt{3}\right)g_{n+v}\right]$$

$$y_{n+w} = y_n + \frac{h}{4320}\left[\left(576+280\sqrt{3}\right)f_{n+u} + 1008f_{n+w} + \left(576-280\sqrt{3}\right)f_{n+v}\right]$$
$$+\frac{h^2}{4320}\left[\left(10+12\sqrt{3}\right)g_n - 140g_{n+w} + \left(10-12\sqrt{3}\right)g_{n+v}\right]$$

$$y_{n+v} = y_n + \frac{h}{34560}\left[\left(4608+2636\sqrt{3}\right)f_{n+u} + \left(8064+4608\sqrt{3}\right)f_{n+w} + \left(4608+1396\sqrt{3}\right)f_{n+v}\right]$$
$$+\frac{h^2}{34560}\left[\left(161+96\sqrt{3}\right)g_n - 40g_{n+w} - \left(271+96\sqrt{3}\right)g_{n+v}\right] \qquad (22)$$

Solving the block hybrid discrete scheme simultaneously, we obtain the higher order implicit second-derivative Runge-Kutta collocation method and writing in the form of (15) as follows:

$$y_n = y_{n-1} + h\left(\frac{4}{15}\right)F_1 + h\left(\frac{7}{15}\right)F_2 + \left(\frac{4}{15}\right)F_3 + h^2\left(\frac{\sqrt{3}}{180}\right)G_1 - h^2\left(\frac{\sqrt{3}}{180}\right)G_2 \qquad (22a)$$

where the internal stage values at the n^{th} step are computed as:

$$Y_1 = y_{n-1},$$

$$Y_2 = y_{n-1} + h\left(\frac{2}{15} - \frac{349\sqrt{3}}{8640}\right)F_1 + h\left(\frac{7}{30} - \frac{2\sqrt{3}}{15}\right)F_2 + h\left(\frac{2}{15} - \frac{659\sqrt{3}}{8640}\right)F_3$$
$$-h^2\left(\frac{271}{34560} - \frac{\sqrt{3}}{360}\right)G_1 - h^2\left(\frac{1}{180}\right)G_2 + h^2\left(\frac{161}{34560} - \frac{\sqrt{3}}{360}\right)G_3$$

$$Y_3 = y_{n-1} + h\left(\frac{2}{15} + \frac{7\sqrt{3}}{108}\right)F_1 + h\left(\frac{7}{30}\right)F_2 + h\left(\frac{2}{15} - \frac{7\sqrt{3}}{108}\right)F_3$$
$$+h^2\left(\frac{1}{432} + \frac{\sqrt{3}}{360}\right)G_1 - h^2\left(\frac{7}{216}\right)G_2 + h^2\left(\frac{1}{432} - \frac{\sqrt{3}}{360}\right)G_3$$

$$Y_4 = y_{n-1} + h\left(\frac{2}{15} + \frac{659\sqrt{3}}{8649}\right)F_1 + h\left(\frac{7}{30} + \frac{2\sqrt{3}}{15}\right)F_2 + h\left(\frac{2}{15} + \frac{349\sqrt{3}}{8640}\right)F_3$$
$$+ h^2\left(\frac{161}{34560} + \frac{\sqrt{3}}{360}\right)G_1 - h^2\left(\frac{1}{864}\right)G_2 - h^2\left(\frac{271}{34560} + \frac{\sqrt{3}}{360}\right)$$

$$Y_5 = y_{n-1} + h\left(\frac{4}{15}\right)F_1 + h\left(\frac{7}{15}\right)F_2 + \left(\frac{4}{15}\right)F_3 + h^2\left(\frac{\sqrt{3}}{180}\right)G_1 - h^2\left(\frac{\sqrt{3}}{180}\right)G_2$$

where the stage derivatives are calculated as follows:

$$\begin{aligned}
F_1 &= f(x_{n-1} + h(0), Y_1),\\
F_2 &= f\left(x_{n-1} + h\left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right), Y_2\right),\\
F_3 &= f\left(x_{n-1} + h\left(\frac{1}{2}\right), Y_3\right),\\
F_4 &= f\left(x_{n-1} + h\left(\frac{1}{2} + \frac{\sqrt{3}}{4}\right), Y_4\right),\\
F_5 &= f(x_{n-1} + h(1), Y_5).
\end{aligned}$$

The implicit second-derivative Runge-Kutta collocation method has order p = 6. Writing the method in an extended Butcher Tableau (16), we have

$\frac{2-\sqrt{3}}{4}$	$\frac{4608-1396\sqrt{3}}{34560}$	$\frac{8064-4608\sqrt{3}}{34560}$	$\frac{4608-2636\sqrt{3}}{34560}$	0	$\frac{-(271-96\sqrt{3})}{34560}$	$\frac{-40}{34560}$	$\frac{161-96\sqrt{3}}{34560}$	0
$\frac{1}{2}$	$\frac{576+280\sqrt{3}}{4320}$	$\frac{1008}{4320}$	$\frac{576-280\sqrt{3}}{4320}$	0	$\frac{10+12\sqrt{3}}{4320}$	$\frac{-140}{4320}$	$\frac{10-12\sqrt{3}}{4320}$	0
$\frac{2+\sqrt{3}}{4}$	$\frac{4608+2636\sqrt{3}}{34560}$	$\frac{8064+4608\sqrt{3}}{34560}$	$\frac{4608+1396\sqrt{3}}{34560}$	0	$\frac{161+96\sqrt{3}}{34560}$	$\frac{-40}{34560}$	$\frac{-(271+96\sqrt{3})}{34560}$	0
1	$\frac{48}{180}$	$\frac{84}{180}$	$\frac{48}{180}$	0	$\frac{\sqrt{3}}{180}$	0	$\frac{-\sqrt{3}}{180}$	0
	$\frac{48}{180}$	$\frac{84}{180}$	$\frac{48}{180}$	0	$\frac{\sqrt{3}}{180}$	0	$-\frac{\sqrt{3}}{180}$	0

3. ANALYSIS OF THE IMPLICIT SECOND-DERIVATIVE RUNGE-KUTTA COLLOCATION METHODS

3.1 Order, Consistency, Zero-stability and Convergence of SDRKC Methods

With the multistep collocation formula (2) we associate the linear difference operator ℓ defined by

$$\ell[y(x);h] = \sum_{j=0}^{r} \alpha_j(x) y(x+jh) + h\sum_{j=0}^{s} \beta_j(x) y'(x+jh) + h^2 \sum_{j=0}^{t} \gamma_j(x) y''(x+jh) \tag{22b}$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a,b]$, following Yakubu [2010], we can write the terms in (20b) as a Taylor series expansion about the point x to obtain the expression,

$$\ell[y(x);h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_p h^p y^{(p)}(x) + \cdots$$

Where the constant coefficients C_p, $p = 0, 1, 2, \ldots$ are given as follows:

$$C_{0:} = \sum_{j=0}^{r} \alpha_j$$

$$C_1 = \sum_{j=1}^{r} j\alpha_j$$

$$C_2 = \sum_{j=1}^{r} j\alpha_j - \sum_{j=0}^{s} \beta_j$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$C_p = \frac{1}{p!}\left(\sum_{j=0}^{r} j\alpha_j - \frac{2}{(p-1)!}\sum_{j=1}^{s} j^{p-1}\beta_j - \frac{1}{(p-2)!}\sum_{j=0}^{t} j^{p-2}\gamma_j\right), p = 3,4,\cdots$$

According to Yakubu [2010], the multistep collocation formula (2) has order p if

$$\ell[y(x); h] = O\left(h^{(p+1)}\right), C_0 = C_1 = \cdots C_p = 0, C_{p+1} \neq 0.$$

Therefore C_{p+1} is the error constant and $C_{p+1}h^{p+1}y^{(p+1)}$ is the principal local truncation error at the point x_n (Chan and Tai [2010]). Therefore, the order and the error constants for the two methods constructed are represented in the Tables below.

TABLE 1: Order and error constants of SDRK collocation methods
Order and Error Constants of the ISDRK method for (19)

Order (p)	Error Constant (C_{P+1})
5	$\frac{-7}{1920}$
5	$\frac{-11}{320} + \frac{\sqrt{2}}{60}$
5	$\frac{-11}{320} - \frac{\sqrt{2}}{60}$

TABLE 2: Order and error constants of SDRK collocation methods
Order and Error Constants of the ISDRK method for (21)

Order (p)	Error Constant (C_{P+1})
6	$\frac{17}{143360}$
6	$\frac{51-27\sqrt{3}}{4480}$
6	$\frac{51}{35840}$
6	$\frac{471}{4480} + \frac{809\sqrt{3}}{57344}$

Definition 3.1. According to Yakubu and Kwami [2015], The implicit second-derivative Runge-Kutta collocation (18) and (20) are said to be consistent if the order of the individual method is greater than or equal to one, that is, if $p \geq 1$.

$(i)\quad \rho(1) = 0$ *and*

$(ii)\quad \rho'(1) = \sigma(1)$, *where* $\rho(z)$ *and* $\sigma(z)$ *are respectively the* 1*st and* 2*nd characteristic polynomials.*

Definition 3.2. According to Yakubu *et,al.* [2010], The second derivative Runge-Kutta collocation methods (18) and (20) are said to be zero-stable if the roots

$$\rho(\lambda) = \det\left[\sum_{i=0}^{k} A^i \lambda^{k-1}\right] = 0$$

Satisfies $|\lambda_j| \leq 1,\ j = 1, 2, \ldots, k$ and for those roots with $|\lambda_j| = 1$, the multiplicity does not exceed 2.

Definition 3.3. According to Yakubu *et,al.* [2010], The necessary and sufficient conditions for the SDRK collocation methods (18) and (20) to be convergent are that they must be consistent and zero-stable. Hence, our methods are convergent.

3.2. Stability regions of the SDRK collocation methods

In this paper, the stability properties of the methods are discussed by reformulating the block hybrid discrete schemes as general linear methods by Butcher [2014] and Butcher and Hojjati [2005]. Hence, we use the notations introduced by Butcher and Hojjati [2005], where a general linear method is represented by a partitioned $(s+r)x(s+r)$matrix (containing A,U, B and V),

$$\begin{bmatrix} Y^{[n]} \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \begin{bmatrix} hf\left(Y^{[n]}\right) \\ y^{[n]} \end{bmatrix}, \quad n = 1,2,\ldots,N \tag{23}$$

Where

$$Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \vdots \\ Y_s^{[n]} \end{bmatrix}, \quad y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ \vdots \\ y_r^{[n-1]} \end{bmatrix}, \quad f\left(Y^{[n]}\right) = \begin{bmatrix} f\left(Y_1^{[n]}\right) \\ f\left(Y_2^{[n]}\right) \\ \vdots \\ f\left(Y_s^{[n]}\right) \end{bmatrix}, \quad y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 \\ A & B \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & u & c-u \end{bmatrix}, \quad B = \begin{bmatrix} A & B \\ 0 & 0 \\ v^T & \omega^T \end{bmatrix}, \quad V = \begin{bmatrix} I & \mu & e-\mu \\ 0 & 0 & I \\ 0 & 0 & I-\theta \end{bmatrix},$$

and $e = [1,\ldots,1]$.

Hence (21) takes the form

$$\begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \vdots \\ Y_s^{[n]} \\ ---- \\ y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix} = \begin{bmatrix} A & U \\ --- & \\ B & V \end{bmatrix} \begin{bmatrix} hf\left(Y_1^{[n]}\right) \\ hf\left(Y_2^{[n]}\right) \\ \vdots \\ hf\left(Y_s^{[n]}\right) \\ ---- \\ y_1^{[n-1]} \\ y_2^{[n-1]} \\ \vdots \\ y_r^{[n-1]} \end{bmatrix} \qquad (24)$$

Where r denotes quantities as output from each step and input to the next step and s denotes stage values used in the computation of the step $y_1, y_2, \ldots, y_s$. The coefficients of these matrices $A, U, B,$ *and* V indicate the relationship between the various numerical quantities that arise in the computation of stability regions. The elements of the matrices $A, U, B,$ *and* V are substituted into the stability matrix which leads to the recurrent equation

$$y^{[n-1]} = M(z)y^{[n]}, n = 1,2,3,\ldots, N-1, Z = \lambda h$$

where the stability matrix

$$M(z) = V + zB(I - zA)^{-1}U$$

and the stability polynomial of the method can easily be obtained as follows:

$$\rho(\eta, z) = \det\left(r\left(A - Uz - Vz^2\right) - B\right).$$

The absolute stability region of the method is defined as

$$\Re = x \in C : \rho(\eta, z) = 1 \Rightarrow |\eta| \le 1.$$

Computing the stability functions gives the stability polynomials of the methods as follows:

$$R(z) = \frac{\frac{9}{40} z r^2 - \frac{16979}{462400} z^2 r^2 - \frac{29}{40} z r^3 + \frac{1157}{54400} z^2 r^3 + r^2 - \frac{131}{544} z\sqrt{2} r^2 + \frac{99}{1700} r^2 z^2 \sqrt{2} - r^3 - \frac{37}{2720} z\sqrt{2} r^3 - \frac{309}{5440} r^3 z^2 \sqrt{2}}{\frac{9}{40} r^2 - \frac{16979}{231200} z r^2 - \frac{29}{40} r^3 + \frac{1157}{27200} z r^3 - \frac{131}{544}\sqrt{2} r^2 + \frac{99}{850} r^2 z\sqrt{2} - \frac{37}{2720}\sqrt{2} r^3 - \frac{309}{2720} r^3 z\sqrt{2}} \qquad (25a)$$

and

$$R(z) = \frac{\begin{gathered} -\frac{2}{15} r^3 z + r^4 + \frac{19}{30} r^4 z + \frac{67}{4320} r^4 z^2 + \frac{109}{51840} r^4 z^3 - r^3 - \frac{445}{41472} r^3 z^3 \sqrt{3} + \frac{803}{8640} r^3 z\sqrt{3} - \frac{1}{3840} r^3 z^4 \sqrt{3} \\ -\frac{37}{1920} r^3 z^2 \sqrt{3} + \frac{283}{4320} r^3 z^2 + \frac{5}{768} r^3 z^3 + \frac{7}{17280} r^3 z^4 - \frac{299}{8640} r^4 z^2 \sqrt{3} - \frac{1}{3240} r^4 z^3 \sqrt{3} - \frac{803}{8640} r^4 z\sqrt{3} - \frac{1}{51840} r^4 z^4 \sqrt{3} \end{gathered}}{\begin{gathered} -\frac{2}{15} r^3 + \frac{19}{30} r^4 + \frac{67}{2160} r^4 + \frac{109}{17280} r^4 z^2 - \frac{445}{13824} r^3 z^2 \sqrt{3} + \frac{803}{8640} r^3 \sqrt{3} - \frac{1}{960} r^3 z^3 \sqrt{3} - \frac{37}{960} r^3 z\sqrt{3} \\ + \frac{283}{2160} r^3 z + \frac{5}{256} r^3 z^2 + \frac{7}{4320} r^3 z^3 - \frac{299}{4320} r^4 z\sqrt{3} - \frac{1}{1080} r^4 z^2 \sqrt{3} - \frac{803}{8640} r^4 \sqrt{3} - \frac{1}{12960} r^4 z^3 \sqrt{3} \end{gathered}} \qquad (25b)$$

which are plotted to produce the required graphs of the absolute stability regions of the methods as displayed in Figures below:

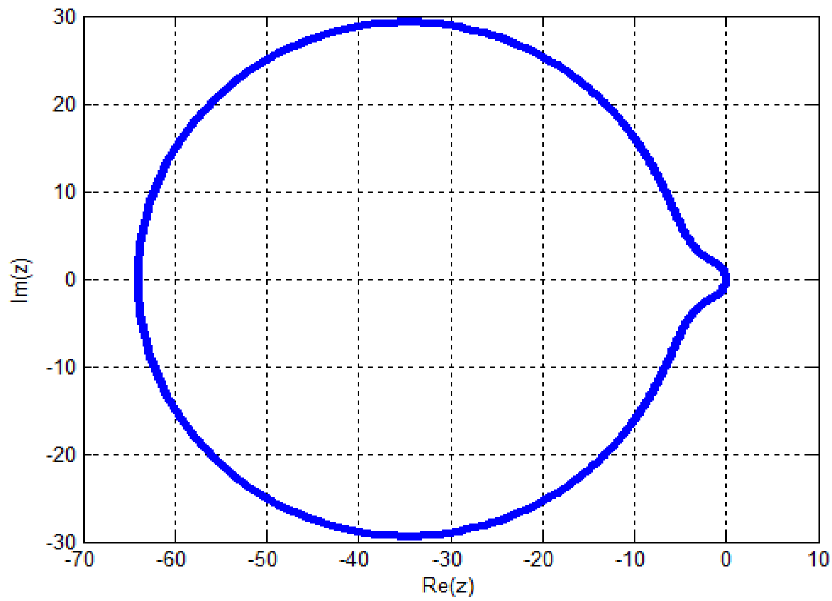

FIGURE 1: Regions of absolute stability of method (19)

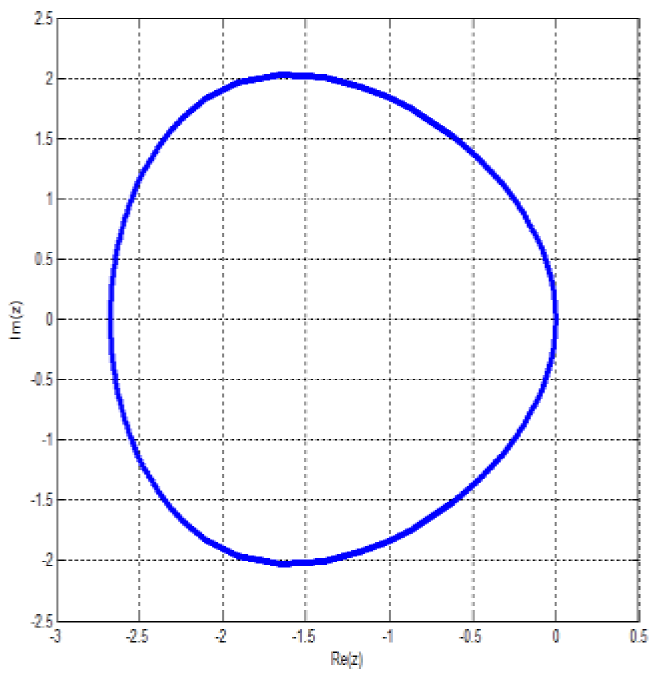

FIGURE 2: Regions of absolute stability of method (21)

Remark: 3.4. The stability regions contain the entire left half complex plane. Clearly, the methods are L-stable, therefore, the methods are A-stable and in addition equations (25a) and (25b) satisfy $\lim_{z\to\infty} R(z) = 0$.

4. NUMERICAL EXPERIMENTS

In this section, we test the effectiveness and validity of the newly derived methods (19) and (21) by applying to some highly stiff systems of initial value problems of the form (1).

Preliminary numerical experiments have been carried out using a constant step size implementation in Matlab. The test examples are some systems of ordinary differential equations written as first order initial value problems. We solved these systems and compared the obtained results side by side in Tables.

Example 1:

We consider a nonlinear stiff problem (Chemostat and Micro Organism Culturing Problem)

$$y_1' = -1.690372243y_1 - 0.001190476190y_2$$
$$y_2' = 101.7684513y_1$$
$$y_1(0) = y_2(0) = 0.1 \quad 0 \le x \le 1000,\ h = 0.1 \ \textit{with Stiffness ratio of } 10^3$$

Chemostat is a vessel into which nutrients are pumped to feed a micro organism with limited Concentration of micro in the vessel.

Example 2:

The second test problem is highly Stiff Linear Systems

$$y_1' = -29998y_1 - 59994y_2$$
$$y_2' = 9999y_1 + 19997y_2$$

Exact

$$y_1(x) = (1/9999)(29997e^{-10000x} - 19998e^{-x}) \quad y_1(0) = 1$$
$$y_2(x) = -e^{-10000x} + e^{-x} \quad y_2(0) = 0$$

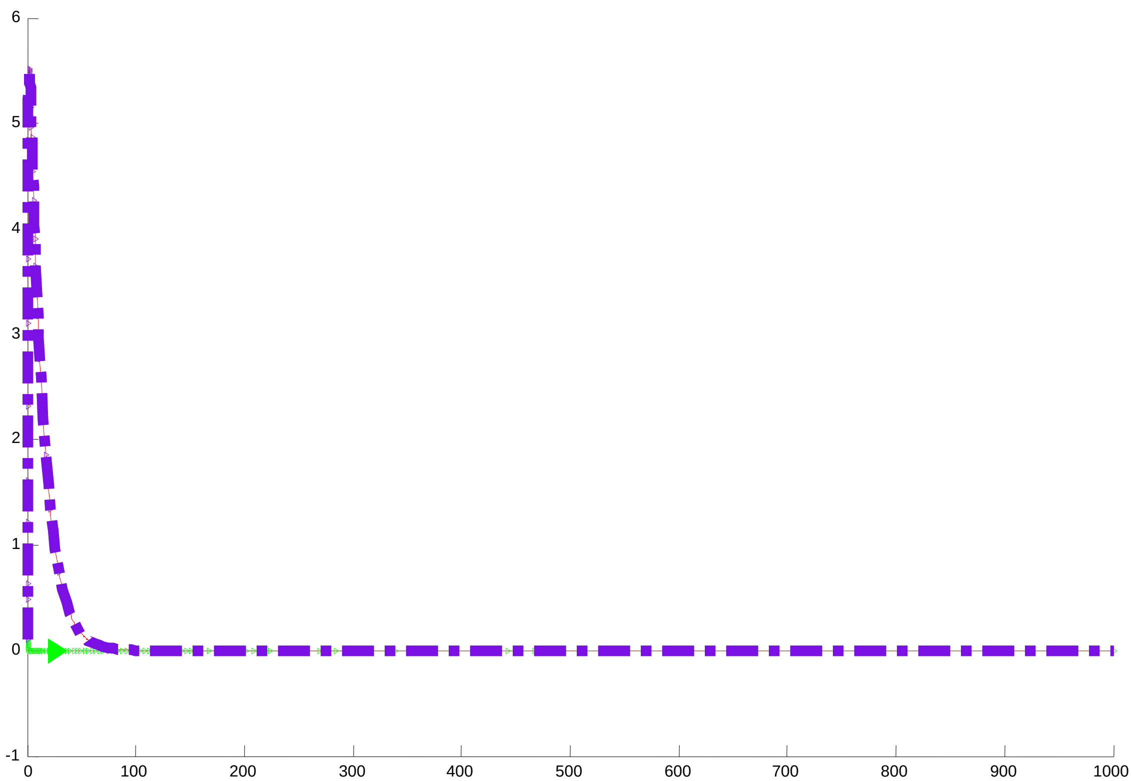

FIGURE 3: Solution Curve of Problem 1 Solved with ISDRK method (19) of order 5

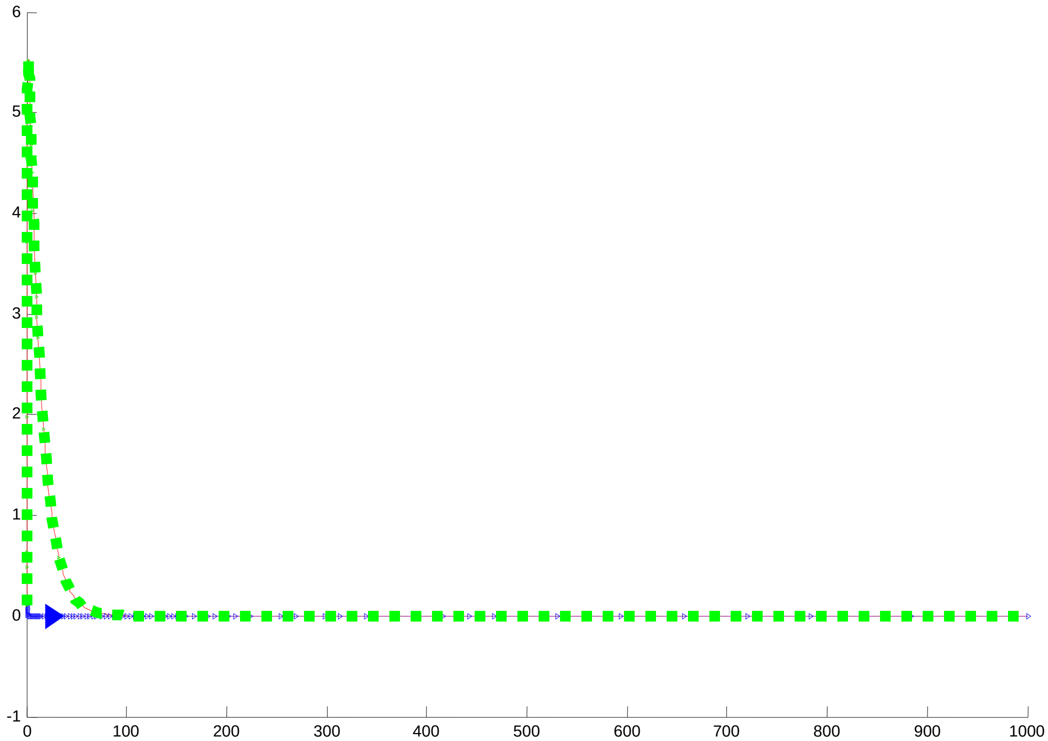

FIGURE 4: Solution Curve of Problem 1 Solved with ISDRK method (21) of order 6

TABLE 3: Absolute error of problem 2 solved with ISDRK method (19) of order 5

h=0.1	Exact Solution		Numerical Solution		Absolute Error(y'-yn')	
x	y1	y2	Y1	y2	y1	y2
20	-2.7067e-001	1.3534e-001	-2.7156e-001	1.3578e-001	8.9265e-004	4.4633e-004
40	-3.6631e-002	1.8316e-002	-3.6873e-002	1.8437e-002	2.4201e-004	1.2101e-004
60	-4.9575e-003	2.4788e-003	-5.0067e-003	2.5034e-003	4.9210e-005	2.4605e-005
100	-9.0800e-005	4.5400e-005	-9.2307e-005	4.6154e-005	1.5072e-006	7.5358e-007
200	-4.1223e-009	2.0612e-009	-4.2603e-009	2.1301e-009	1.3799e-010	6.8993e-011
240	-7.5503e-011	3.7751e-011	-7.8546e-011	3.9273e-011	3.0428e-012	1.5214e-012
260	-1.0218e-011	5.1091e-012	-1.0665e-011	5.3325e-012	4.4686e-013	2.2343e-013
300	-1.8715e-013	9.3576e-014	-1.9663e-013	9.8314e-014	9.4750e-015	4.7375e-015

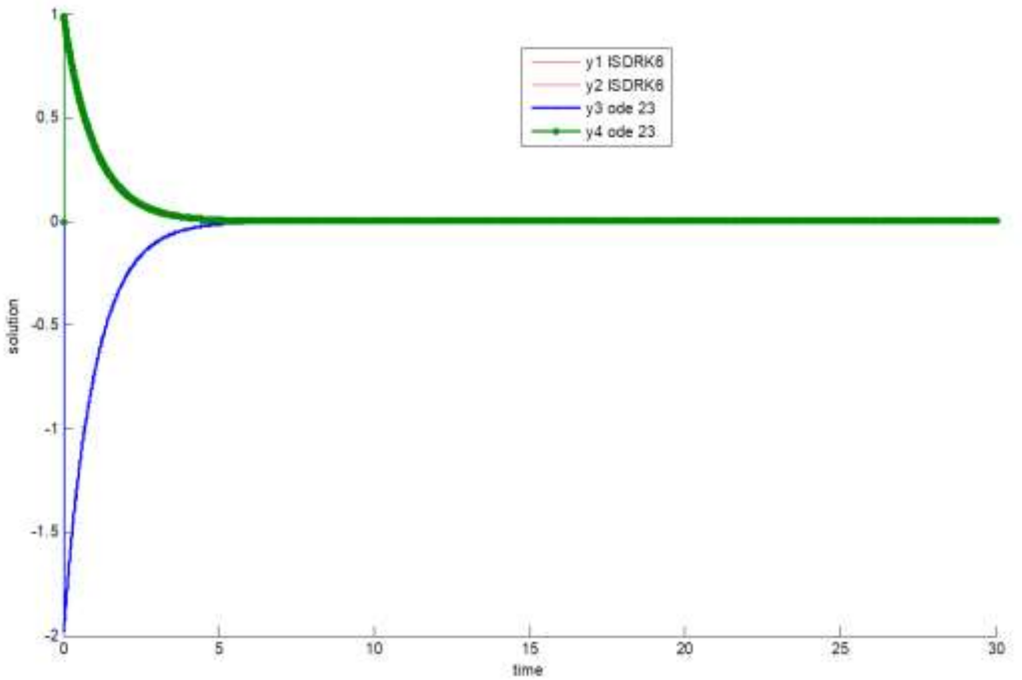

FIGURE 5: Solution Curve of Problem 2 Solved with ISDRK method (19) of order 5

TABLE 4: Absolute error of problem 2 solved with ISDRK method (21) of order 6

h=0.1 x	Exact Solution		Numerical Solution		Absolute Error(y'-yn')	
	y1	y2	y1	y2	y1	y2
20	-1.6015e-15	8.0074e-16	-1.1864e-15	5.9320e-16	6.3825e-22	3.1912e-22
40	-1.3112e-15	6.5559e-16	-9.7134e-16	4.8567e-16	5.2553e-22	2.6277e-22
60	-1.0735e-15	5.3675e-16	-7.9527e-16	3.9763e-16	4.3271e-22	2.1636e-22
100	-7.1959e-16	3.5979e-16	-5.3308e-16	2.6654e-16	2.9333e-22	1.4666e-22
200	-2.6472e-16	1.3236e-16	-1.9611e-16	9.8055e-17	1.1092e-22	5.5459e-23
300	-9.7386e-17	4.8693e-17	-7.2145e-17	3.6073e-17	4.1911e-23	2.0956e-23

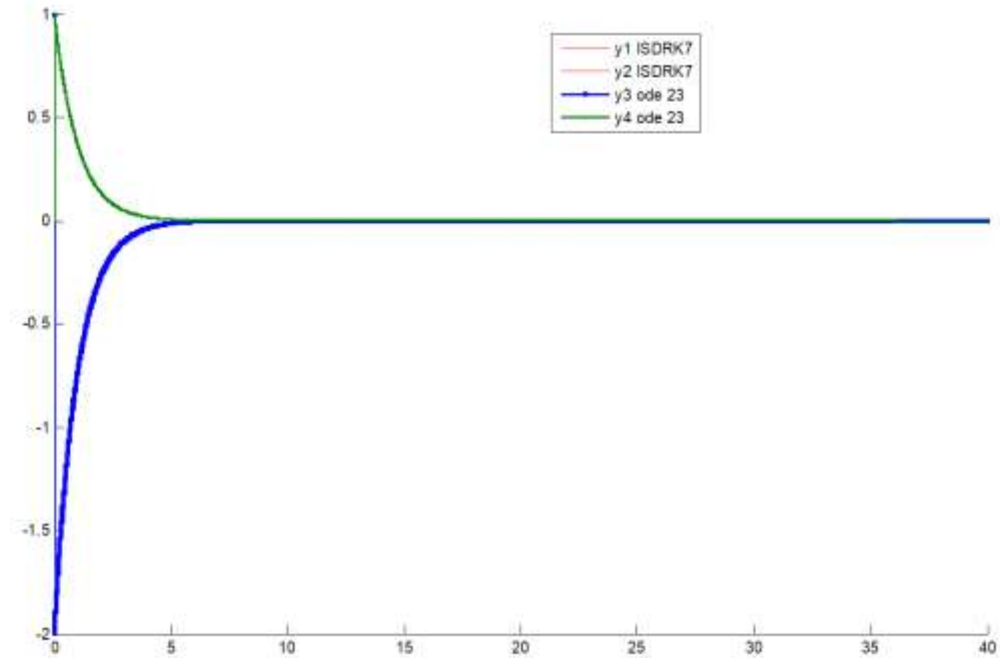

FIGURE 6: Solution Curve of Problem 2 Solved with ISDRK method (21) of order 6

CONCLUSION

The purpose of the present paper has been to introduce a special class of implicit second-derivative Runge-Kutta collocation methods suitable for the approximate numerical integration of systems of ordinary differential equations. The derived methods provide an efficient way to find numerical solutions to systems of initial value problems when the second derivative terms are cheap to evaluate. We present two new methods of orders five and six. We also presented summary of numerical comparisons between the new methods on a set of two systems of initial value problems. Chemostat and Micro Organism Culturing Model. This model is often used to study the specific microbial interactions or parameters associated with infection and disease. From the solution curves we observed that there is a steady culture of the Micro Organism and our method was able to maintain the steady culture just like the ode 23. The numerical comparisons as well as establishing the efficiency of the new methods show that the order five method and the order six method shows more accuracy on all the problems considered in the paper.

ACKNOWLEDGMENTS

The author is indebted to the referees for their helpful suggestions and contributions to the completion of this paper.

REFERENCES

[1] Adesanya, A. O., Fotta, A.U., & Onsachi, R.O. (2016), Runge-Kutta Collocation Method for the Solution of First Order Ordinary Differential Equations. *Journal of Nonlinear Analysis and Differential Equations,* Vol. 4, no. 1, 17-26.

[2] Akinfenwa, O.A., Abdulganiy, R.I., Akinnukawe, B.I., Okunuga, S.A. &Rufai,U.O. (2017), Continuous L-Stable Multi-Derivative Hybrid Implicit Runge-Kutta Method of Order Five for the Integration of Stiff Problems, *International Journal of Management and Applied Science*. Volume-3, Issue1. ISSN: 2394-7926.

[3] Butcher JC. Order and stability of Runge–Kutta methods, Nanjing University Seminars, May, 2014.

[4] Butcher, J.C. &Hojjati, G. (2005), Second Derivative Methods with Runge-Kutta Stability, *Numerical Algorithms*, 40 415 – 429.

[5] Chan, R. P.K. & Tsai, A. Y. J. (2010), On Explicit Two-derivative Runge-Kutta Methods, *NumerAlgor* 53,171-194.

[6] Donald,J.Z., Skwame, Y.,& Dominic, R. (2015), Reformulation of implicit one-step Legendre polynomial Hybrid Block Method in form of implicit Runge-Kutta collocation methods for the solutions of first-order ordinary differential equations.*ADSUJSR*, 03(2):356-367.

[7] Markus, S. &Yakubu, D.G. (2016), Application of Second Derivative Runge-Kutta Collocation Methodsto Stiff Systems of Initial Value Problems*. Leonard Journal of Science*. Vol-3, 43-66

[8] Yakubu, D. G. (2003), Single-step Stable Implicit Runge-Kutta Method Based on Lobatto Points for Ordinary Differential Equations. *Journal of the Nigerian Mathematical Society,* 22, 57-70.

[9] Yakubu, D.G., Hamza, A. Markus, S., Kwami, A.M. &Tumba, P. (2010), Uniformly Accurate Order Five Radau-Runge-Kutta Collocation Methods. *Abacus Journal of Mathematical Association of Nigeria, Mathematics series,* 372A75-94.

[10] Yakubu, D.G., Manjak, N. H., Buba S.S., &Maksha, A. I. (2011), A Family of Uniformly Accurate Order Runge-Kutta Collocation Methods, *Journal of Computational and Applied Mathematics,* Vol.30, No.2, Pp 315-330.

[11] Yakubu, D.G.&Kwami, A.M. (2015), Implicit Two- Derivative Runge-Kutta Collocation Methods for Systems of Initial Value Problems. *Journal of the Nigerian Mathematical Society. 34(2):128-142.*

[12] Yakubu, D.G., Kumleng, G.M., and Markus, S.(2017), Second derivative Runge-Kutta Collocation Methods based on Lobatto nodes for stiff systems. *Journal of modern methods in numerical mathematics 8:1-2, page 118-138.*

[13] Yusuf, S., G.M Kumleng & Donald J.Z. (2019), Implicit Second-Derivative Runge-Kutta Collocation Methods of Uniformly Accurate Order 3 and 4 for the Solution of Systems of initial Value Problems